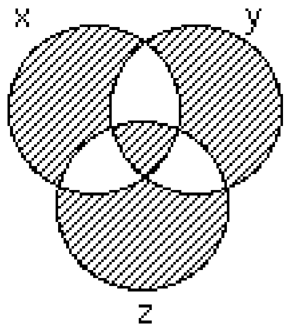

Commutativity: $x \Delta y = x \setminus y \cup y \setminus x = y \setminus x \cup x \setminus y = y \Delta x$.

(b) $*$ on $\mathbf{R} \setminus \{1\}$ defined by $a * b = a + b - ab$ for all $a, b \in \mathbf{R} \setminus \{1\}$.

If $a, b \neq 1$, then $a + b - ab \in \mathbf{R}$, but if $a + b - ab = 1$, then $a(1-b) + b = 1$, $a(1-b) = 1 - b$, which is impossible since $a, b \neq 1$. So $*$ is a binary operation on $\mathbf{R} \setminus \{1\}$.

Commutativity: $\forall a, b \in \mathbf{R} \setminus \{1\}$, $a * b = a + b - ab = b + a - ba = b * a$.

Associativity: $\forall a, b, c \in \mathbf{R} \setminus \{1\}$, $a * (b * c) = a + (b * c) - a(b * c)$
$= a + b + c - bc - a(b + c - bc) = a + b + c - ab - ac - bc + abc$. Similarly,
$(a * b) * c = a + b + c - ab - ac - bc + abc$. So $a * (b * c) = (a * b) * c$.

4. (a) Prove by induction that for each $n \in \mathbf{N}$, $4^n - 1$ is divisible by 3.

Let P_n be the statement: $4^n - 1$ is divisible by 3.

P_1 is true since $4^1 - 1 = 3 = 3 \cdot 1$.

Assume $k \geq 1$ and $4^k - 1 = 3r$. Then $4^{k+1} - 1 = 4(4^k) - 1 = 4(3r + 1) - 1 = 3(4r + 1)$ which is divisible by 3. Hence $P_k \Rightarrow P_{k+1}$ is true for all $k \geq 1$, So by the PMI, P_n is true for all $n \in \mathbf{N}$.

(b) Criticise the following:

Theorem: We are given n coins, where n is an integer > 1. All but one of the coins are the same weight and the other is heavier. We have a balance. Then 4 weighings suffice to discover which coin is heavier.

Proof (By induction.)

When n=2 the result is clear. Suppose we have proved the result for k coins. We are now given k+1 coins. We proceed as follows. Set one coin aside. Apply the procedure for k coins to the remaining k coins. If we find the heavy coin then we are finished. If not, then the heavy coin is the one we set aside. Thus we have a procedure for k+1 coins. The theorem follows by induction.

If P_n is the statement: "Given a set of n coins all the same weight except for one which is heavier, it is possible with at most 4 weighings on a balance to discover the heavy one," then, as indicated, P_1 is clearly true. But the method proposed for showing that $P_k \Rightarrow P_{k+1}$ is true fails because when one coin is removed from the set of $k + 1$ coins, the condition that the remaining coins are all the same except for one heavy one is not met, so the induction hypothesis does not apply, and the conclusion that 4 weighings is sufficient is not valid.

DEPARTMENT OF MATHEMATICS

MATHS 255	Assignment 4 Solutions	Due: 15 August, 2001

1. (a) Prove or give a counterexample: If $f: X \to Y$ is a function and C,D are disjoint subsets of Y then $f^{-1}(C)$, $f^{-1}(D)$ are disjoint subsets of X.

Proof: Suppose C,D are disjoint subsets of Y and $x \in f^{-1}(C) \cap f^{-1}(D)$. Then $f(x) \in C$ since $x \in f^{-1}(C)$ and $f(x) \in D$ since $x \in f^{-1}(D)$. hence $f(x) \in C \cap D$, contradicting our asumption that C,D are disjoint.

(b) Prove or give a counterexample: If $f: X \to Y$ is a function and A,B are disjoint subsets of X then $f(A)$, $f(B)$ are disjoint subsets of Y.

Counterexample: We need to specify sets X, Y, A, B, and a function f. So for example let $X = Y = \{a,b,c\}$, (distinct elements), let $f: X \to Y$ be defined by $f(a) = a$, $f(b) = b$, $f(c) = b$. Let $A = \{a,b\}$, $B = \{c\}$. Then A,B are disjoint, but $b \in f^{-1}(A) \cap f^{-1}(B)$.

2. Show that if u is a subsequence of t and t is a subsequence of s, then u is a subsequence of s. [Use the definition of subsequence in the text, and prove what you need to prove about composition of increasing functions.]

There is a strictly increasing function $n: \mathbf{N} \to \mathbf{N}$ such that $u(i) = t(n(i))$ for all i, and there is a strictly increasing function $m: \mathbf{N} \to \mathbf{N}$ such that $t(i) = s(m(i))$ for all i. Hence, $u(i) = t(n(i)) = s(m(n(i)))$ for all i. Since m,n are strictly increasing, so also is $m \circ n$ (proof below). Hence u is a subsequence of s.

Fact: The composition of two strictly increasing functions is strictly increasing. Proof: Suppose m,n are strictly increasing functions from a poset A to itself. Assume $i, j \in A$ with $i < j$. Then $n(i) < n(j)$ since n is strictly increasing, and $m(n(i)) < m(n(j))$ since m is strictly increasing. In other words, $m \circ n(i) < m \circ n(j)$. So $m \circ n$ is strictly increasing as required.

3. Determine which of the following are binary operations, and for those which are, determine whether they are associative, commutative.

(a) Symmetric difference on the set of all *finite* subsets of an *infinite* set A.

Let S be the set of finite subsets of A. For all $x, y \in S$, $x \Delta y \subseteq x \cup y$, If x,y have m,n elements respectively, then $x \cup y$ has at most $m + n$ elements, and so is finite. Hence $x \Delta y \in S$. So Δ is a binary operation on S. To show associativity, we might use either a Venn diagram or truth tables. We try a Venn diagram to show that $\forall x, y, z \in S$, $(x \Delta y) \Delta z = x \Delta (y \Delta z)$. By careful drawing and shading, we find that both sides are the following: