DEPARTMENT OF MATHEMATICS

MATHS 255	Assignment 3 Solutions	Due: 8 August, 2001

1. Suppose that X is a poset with partial ordering $\leq$, and suppose that A is a non-empty subset of X. Show that if A has a least upper bound and a greatest lower bound, then $\text{glb}(A) \leq \text{lub}(A)$.

   Let L,l be least upper and greatest lower bounds respectively for A. Since $A \neq \emptyset$, let $a \in A$. Then $l \leq a$ since l is a lower bound of A, and $a \leq L$ since L is an upper bound of A. Hence, $l \leq L$ by transitivity.

2. Let $A = \mathbf{N} \times \mathbf{N}$, and define a relation ~ on A by $(a,b) \sim (c,d) \iff b + c = a + d$. Prove that ~ is an equivalence relation on A, and describe the equivalence classes.

   [Note: This set of equivalence classes, endowed with appropriate definitions of addition and multiplication, is sometimes called the set of *integers*.]

   1. Reflexive: Let $(a,b) \in \mathbf{N} \times \mathbf{N}$. Then $a + b = b + a$, so $(a,b) \sim (a,b)$.

   2. Symmetric: Let $(a,b),(c,d) \in \mathbf{N} \times \mathbf{N}$, and suppose $(a,b) \sim (c,d)$. Then $b + c = a + d$, so $d + a = c + b$, and hence $(c,d) \sim (a,b)$.

   3. Transitive: Let $(a,b),(c,d),(e,f) \in \mathbf{N} \times \mathbf{N}$, and suppose $(a,b) \sim (c,d)$ and $(c,d) \sim (e,f)$. Then $a + d = b + c$, and $c + f = d + e$. Adding the two equations, we get $a + d + c + f = b + c + d + e$, and hence $a + f = b + e$, so that $(a,b) \sim (e,f)$.

   $T_{(a,b)} = \{(x,y) : x,y \in \mathbf{N} \text{ and } y = x + (b - a)\}$ = all points of the plane with natural number coordinates and on the line through (a,b) parallel to $y = x$. [Naturally, we identify the class containing (a,b) with the integer $b - a$.]

3. (a) Prove that if functions $f: A \to B$ and $g: B \to C$ are onto, then so is $g \circ f$.

   Let $c \in C$. $\exists b \in B$, $g(b) = c$ (since g is onto), and $\exists a \in A$, $f(a) = b$ (since f is onto), so $g \circ f(a) = c$ as required to show that $g \circ f$ is onto.

(b) Prove that if $f: A \to B$ is a function and the inverse relation f^{-1} from B to A is a function, then f is one-to-one and onto.

$f^{-1} = \{(b,a) : f(a) = b\}$. Suppose $a_1, a_2 \in A$ and $f(a_1) = f(a_2)$. Then $(f(a_1), a_1) \in f^{-1}$ and $(f(a_1), a_2) \in f^{-1}$. Since f^{-1} is single valued, we conclude that $a_1 = a_2$, and hence f is one-to-one.

To show f is onto, assume $b \in B$. Then since B is the domain of f^{-1}, $(b,a) \in f^{-1}$ for some $a \in A$. Hence, $f(a) = b$, as required to show that f is onto.

4. Let $f: A \to B$ be a function, and define a new function $F: \wp(B) \to \wp(A)$ by

$$F(C) = \{a \in A : f(a) \in C\}$$

for each $C \subseteq B$. Prove that f is one-to-one if and only if F is onto.

Assume that f is one-to-one, and suppose $C \in \wp(A)$. Let $D = f(C)$. We show $F(D) = C$. $x \in F(D) \Leftrightarrow f(x) \in D \Leftrightarrow f(x) = f(c)$ for some $c \in C$ $\Leftrightarrow x = c$ for some $c \in C$ (since f is one-to-one) $\Leftrightarrow x \in C$. Hence $F(D) = C$, and F is onto.

Conversely, assume F is onto and $a \in A$. Then $\{a\} \in \wp(A)$, so $\{a\} = F(C)$ for some $C \subseteq B$ (since F is onto). Now $F(C) = \{x \in A : f(x) \in C\}$, so $a \in F(C) \Rightarrow f(a) \in C$. Now if $a' \in A$ and $f(a) = f(a')$, then $f(a') \in C$, so $a' \in F(C) = \{a\}$, and so $a' = a$, and hence f is one-to-one.