DEPARTMENT OF MATHEMATICS

MATHS 255FC — Assignment 10 solutions — Due 10th October 2001

1. Consider $f(x)=\begin{cases} x.\sin\frac{1}{x}, & x\neq 0\\ 0, & x=0\end{cases}$

(a) Show from first principles that *f(x)* is continuous at *x* = 0. (b) Is *f(x)* continuous at other points in **R**?. (c) Is *f(x)* is differentiable at 0? (d) Is *f(x)* differentiable at other points in **R**? Explain.

(a) $|x-0|<\delta \Rightarrow \left|x.\sin\frac{1}{x}-0\right| \le |x| < \varepsilon = \delta$. (b) Continuous for $x\neq 0$ as the function is a composite of standard continuous functions. (c) $\lim_{h\to 0}\frac{f(o+h)-f(0)}{h} = \lim_{h\to 0}\frac{h\sin\frac{1}{h}}{h} = \lim_{h\to 0}\left(\sin\frac{1}{h}\right)$ does not exist.

(d) Differentiable at $x\neq 0$ as the function is a composite of standard differentiable functions.

2. Determine where the following function is continuous and where it is differentiable:

$$f(x)=\begin{cases} x^4-3x^3+2x^2, & x\in\mathbf{Q}\\ 0, & x\notin\mathbf{Q}\end{cases}$$

Since the definitions oscillate between 0 and non zero values, the only possible points where the function $x^4-3x^3+2x^2 = x^2(x-1)(x-2)=0$. I.e. *x* = 0,1, 2. At each of these values both definitions of $f(x)$ tend to zero as *x* tends to 0, 1, 2 so *f* os continuous at these points. At 0 the function is also differentiable since $\lim_{h\to 0}\frac{f(o+h)-f(0)}{h} = \lim_{h\to 0}\frac{h^2(h-1)(h-2)}{h} = \lim_{h\to 0}(h(h-1)(h-2)) = 0$ for *h* rational and equals *0* for *h* irrational. It is not differentiable at 1 or 2 since the derivative of the function $x^4-3x^3+2x^2$ is non zero at these points but the derivative of 0 is zero.

3(a) Suppose that f is continuous and bounded on **R**. Either prove that f attains a maximum or a minimum value on **R**, or give a counter example. $f(x)=\frac{1}{1+e^x}$ and $\tan^{-1}x$ are counterexamples. Each function has two horizontal asymptotes which give a glb and lub which *f(x)* never reaches.

3(b). Suppose that *f* is continuous at every point and that $f(x)\to 0$ as $x\to\pm\infty$. Prove that *f* attains a maximum or a minimum value on **R**.

If *f* = 0, *f* attains its max and min at every point. If *f* ≠ 0, $\exists x_0$: $f(x_0)\neq 0$, suppose $f(x_0)>0$.

Then let $\varepsilon = f(x_0)/2$

$\exists N_+, N_-: x>N_+ \;\; or \;\; x<-N_- \Rightarrow |f(x)|<\varepsilon$. Let $N=\max(N_+,N_-)$ then $|x|>N \Rightarrow |f(x)|<\varepsilon$.

Now let $M_N = \max_{x\in[-N,N]} f(x) = f(x_m)$. Then $M_N = f(x_m)\ge f(x_0)\ge f(x)$ for all *x* outside [-N,N].

Now $f(x_m)=M_N=M=\operatorname{lub}_{x\in(-\infty,\infty)} f(x)$ since $f(x_m)$ is a max on [-N,N] and all values outside are also less than this value. So *f* achieves its max and min on **R**..

4. Let a real valued function *f* be continuous on the closed interval $[a,b]$. Suppose that for each $x\in[a,b]$ there exists there exists a $y\in[a,b]$ such that $|f(y)|\le\frac{1}{2}|f(x)|$. Prove there exists a $z\in[a,b]$ for which $f(z)=0$. (Hint: Use the Extreme Value Theorem).

Either: Consider the function $g(x)=|f(x)|$. Glb *g(x)* = 0 since $g(x)\ge 0$ and $\exists x_n$:$|f(x_n)|<\frac{1}{2^n}|f(a)|$.

Thus by the extreme value theorem, min $|f(x)|=0$ and $\exists c\in[a,b]$:$|f(c)|=0$.

Or: Given $x_1=a$, let x_i be such that $|f(x_i)|\le\frac{1}{2}|f(x_{i-1})|$ then $\{x_i\}$ is a sequence in [*a*,*b*] with $\{f(x_i)\}\le\frac{1}{2^i}|f(a)|\to 0$, but [*a*,*b*] is bounded so there is a convergent subsequence $\{x_{i_k}\}\to c$, with *c* in [*a.b*] since this interval is closed. Since *f* is continuous $\{x_{i_k}\}\to c \Rightarrow \{f(x_{i_k})\}\to f(c)=0$.