MATHS 255 **Assignment 1 Solutions** **Due 25 July 2001**

1. Write the following in symbolic form, using symbols $\sim, \wedge, \vee, \Rightarrow, \Leftrightarrow, \forall, \exists$, letters x,y,z ... for free variables and capital letters A,B,C... for statements.

a. Either Joe is smart or he is lucky but not both.

A: Joe is smart. B: Joe is lucky. $(A \vee B) \wedge \sim (A \wedge B)$.

b. Doing homework regularly is a necessary condition for me to pass this course, but it is not sufficient.

A: I do homework regularly. B: I pass this course. $(B \Rightarrow A) \wedge \sim (A \Rightarrow B)$.

c. It is not the case that if you are either unkind to me or unkind to my friend then I will neither sing to you nor talk to you. (Also rewrite this in a more positive way.)

A: You are kind to me. B: You are kind to my friend. C: I will sing to you. D: I will talk to you. $\sim [(\sim A \vee \sim B) \Rightarrow (\sim C \wedge \sim D)]$. Equivalent to $\sim (A \wedge B) \wedge (C \vee D)$: Even though you have not been kind to both me and my friend, I will sing or talk to you.

d. Every real number is a sum of two distinct real numbers.

$\forall x \in R \;\; \exists y \in R \;\; \exists z \in R \;\; (y \neq z) \;\wedge\; (x = y + z)$.

e. Given any two real numbers, there is a real number which is less than their sum.

$\forall x \in R \;\; \forall y \in R \;\; \exists z \in R \;\; z < x + y$.

2. Assume A,B,C are statements. Construct truth tables for the following pairs of statements. State whether either implies the other and whether or not they are equivalent:

a. $A \wedge (B \vee C)$, $(A \wedge B) \vee (A \wedge C)$.

A	B	C	$B \vee C$	$A \wedge (B \vee C)$	$A \wedge B$	$A \wedge C$	$(A \wedge B) \vee (A \wedge C)$
T	T	T	T	T	T	T	T
T	T	F	T	T	T	F	T
T	F	T	T	T	F	T	T
T	F	F	F	F	F	F	F
F	T	T	T	F	F	F	F
F	T	F	T	F	F	F	F
F	F	T	T	F	F	F	F
F	F	F	F	F	F	F	F

The two statements are equivalent.

b. $A \wedge \sim B$, $\sim(\sim A \vee B)$.

A	B	~B	A∧~B	~A∨B	~(~A∨B)
T	T	F	F	T	F
T	F	T	T	F	T
F	T	F	F	T	F
F	F	T	F	T	F

The two statements are equivalent.

c. $(A \Leftrightarrow B) \wedge (B \Rightarrow \sim C) \wedge C$, $\sim A$.

A	B	C	$A \Leftrightarrow B$	$B \Rightarrow \sim C$	$(A \Leftrightarrow B) \wedge (B \Rightarrow \sim C)$	$(A \Leftrightarrow B) \wedge (B \Rightarrow \sim C) \wedge C$	~A
T	T	T	T	F	F	F	F
T	T	F	T	T	T	T	F
T	F	T	F	T	F	F	F
T	F	F	F	T	F	F	F
F	T	T	F	F	F	F	T
F	T	F	F	T	F	F	T
F	F	T	T	T	T	F	T
F	F	F	T	T	T	T	T

Neither statement implies the other.

3. Let $f: N \to N$ be given by $f(x) = x^3 + 5x$. (N = natural numbers)

a. Use a direct proof to show that if $n < k$ then $f(n) < f(k)$.

Assume $n < k$. That is, $k - n > 0$. To show $f(n) < f(k)$ we show that $f(k) - f(n) > 0$.

$f(k) - f(n) = k^3 + 5k - n^3 - 5n = k^3 - n^3 + 5(k - n) = (k - n)(k^2 + kn + n^2 + 5) > 0$

since $k - n > 0$ and the terms in the second factor are all positive integers.

b. Use a proof by contrapositive to show that if $f(n) < f(k)$ then $n < k$.

Suppose that $n \geq k$, that is, $n - k \geq 0$. We show (as in part (a))that $f(n) - f(k) \geq 0$.

$f(n) - f(k) = n^3 + 5n - k^3 - 5k = n^3 - k^3 + 5(n - k) = (n - k)(n^2 + nk + k^2 + 5) \geq 0$

since $n - k \geq 0$ and the terms in the second factor are all positive integers.

c. Use a proof by contradiction to show that if $f(n) = f(k)$ then $n = k$.

Suppose that $f(n) = f(k)$ and $n \neq k$. Then $0 = n^3 + 5n - k^3 - 5k = (n - k)(n^2 + nk + k^2 + 5)$. Hence $n - k = 0$ (since the terms in the second factor are all positive integers). So $n = k$ and $n \neq k$, a contradiction.

d. Use a proof by cases to prove that $f(n)$ is a multiple of 3 for all $n \in N$. [You may assume that every integer can be written uniquely in the form $3k$, $3k+1$, or $3k+2$.]

Our assumption says that for each $n \in N$ there are integers $r = 0,1,2$ and integer k such that $n = 3k + r$. So $f(n) = f(3k+r) = (3k+r)^3 + 5(3k+r) = 27k^3 + 27k^2 r + 9kr^2 + r^3 + 15k + 5r = 3(9k^3 + 9k^2 r + 3kr^2 + 5k) + (r^3 + 5r) = 3N + (r^3 + 5r)$ for some integer N.

Case 1: r=0. Then $r^3 + 5r = 0$ which is divisible by 3 so that in this case $f(n)$ is divisible by 3.

Case 2: r=1. Then $r^3 + 5r = 6$ which is divisible by 3 so that in this case $f(n)$ is divisible by 3.

Case 3: r=2. Then $r^3 + 5r = 18$ which is divisible by 3 so that in this case $f(n)$ is divisible by 3.

In any case, $f(n)$ is divisible by 3.

4. a. Prove that for any sets X, Y, $X \subseteq Y$ if and only if $P(X) \subseteq P(Y)$.

Assume first that $X \subseteq Y$. We show that $P(X) \subseteq P(Y)$ by showing that every element of $P(X)$ is an element of $P(Y)$. $A \in P(X) \Rightarrow A \subseteq X \Rightarrow A \subseteq Y \Rightarrow A \in P(Y)$.

Conversely, assume that $P(X) \subseteq P(Y)$. We show that $X \subseteq Y$ by showing that every element of X is an element of Y. $x \in X \Rightarrow \{x\} \subseteq X \Rightarrow \{x\} \in P(X) \Rightarrow \{x\} \in P(Y) \Rightarrow \{x\} \subseteq Y \Rightarrow x \in Y$.

b. Prove that for any sets A, B, $(A \setminus B) \cap B = \varnothing$ and $(A \setminus B) \cup B = A \cup B$.

To prove $(A \setminus B) \cap B \subseteq \varnothing$, we assume that $x \in (A \setminus B) \cap B$ and show that a contradiction results. (I.e. an indirect proof.) So assume $x \in (A \setminus B) \cap B$. Then $x \in A \wedge x \notin B \wedge x \in B$, so that $x \notin B \wedge x \in B$, a contradiction. The reverse inclusion $\varnothing \subseteq (A \setminus B) \cap B$ is obvious since the empty set is a subset of every set.

To prove that $(A \setminus B) \cup B = A \cup B$, we show that each is a subset of the other.

$x \in (A \setminus B) \cup B \Rightarrow (x \in A \wedge x \notin B) \vee (x \in B) \Rightarrow (x \in A) \vee (x \in B) \Rightarrow x \in A \cup B$. Hence, $(A \setminus B) \cup B \subseteq A \cup B$.

On the other hand suppose $x \in A \cup B$. We consider two cases. (1) $x \in B$. (2) $x \notin B$.

(1) $x \in B$ and $B \subseteq (A \setminus B) \cup B$, so $x \in (A \setminus B) \cup B$.

(2) $x \notin B$ and $x \in A \cup B$ implies $x \in A$, and hence $x \in (A \setminus B) \subseteq (A \setminus B) \cup B$.

Thus, $A \cup B \subseteq (A \setminus B) \cup B$.