MATHS 255 Class Notes
Chapter 4

Ordered pairs

If $a \in A$ and $b \in B$, then $(a,b) = \{\{a\},\{a,b\}\}$.

Note: $(a,b) = (c,d) \Leftrightarrow a = c \wedge b = d$.

Cartesian Product (or Direct Product)

$A \times B = \{(a,b) \colon a \in A \text{ and } b \in B\}$

Relations

A *relation between* A and B is any subset of
$A \times B$.

A *relation on* A is any subset of $A \times A$.

If $\rho \subseteq A \times B$ is a relation between A and B, then
when we write $a\rho b$, we mean $(a,b) \in \rho$.

A relation ρ on A is:

reflexive if for all $x \in A$, $x\rho x$,

symmetric if for all $x, y \in A$, $x\rho y \Rightarrow y\rho x$,

antisymmetric if
for all $x, y \in A$, $(x\rho y \wedge y\rho x) \Rightarrow y = x$,

transitive if
for all $x, y, z \in A$, $(x\rho y \wedge y\rho z) \Rightarrow x\rho z$.

Exmples:

2. $A = \mathbf{N}$, $x\rho y \Leftrightarrow x + y$ odd.

3. $A = \mathbf{N}$, $x\rho y \Leftrightarrow x + y$ even.

8. $A = \{a,b,c\}$ (distinct elements),
$\rho = \{(a,a),(b,b),(c,c),(b,a),(a,c),(c,a)\}$.

Orderings

A relation ρ on A is a *partial ordering* if it is reflexive, antisymmetric and transitive.

A relation ρ on A is a *total ordering* if it is a partial ordering and

for all $x, y \in A,\ x\rho y \vee y\rho x$.

A *poset* is a partially ordered set.

A lattice diagram is a graphical representation of a finite poset A with vertices representing points of A and and a path upward from a to b if $a\rho b$.

Suppose A is a poset and $\leq$ a partial ordering on A. and $x \in A$.

x is a *maximal* element if

for all $y \in A,\ x \leq y \Rightarrow x = y$,

or in other words, there is no element y of A such that $x < y$.

x is a *largest* element of A if

for all $y \in A,\ y \leq x$.

There can be at most one largest element, but there may be many maximal elements.

Minimal and *smallest* elements are defined analogously.

If $B \subseteq A$, then x is an *upper bound* of B if

for all $y \in B,\ y \leq x$.

(Note, we do not assume $x \in B$.)

B is *bounded above* if it has an upper bound.

If B is bounded above and the set of upper bounds of B has a smallest element l, then l is the *least upper bound (lub)* of B.

Lower bound, bounded below and *greatest lower bound (glb)* are defined analogously.