MATHS 255 Semester 2, 2001

Page and section numbers refer to text *Chapter Zero* by C Shumacher.

p.5 "Open a book of abstract mathematics and you will find sets, relations, functions, proofs by contrapositive and by contradiction, mathematical induction, the close link between equivalence relations and partitions—in short, a core of ideas that every mathematician knows and uses."

§2.1

Sentence: Complete grammatically correct expression.
"How are you doing?"
"x+y = z."

Predicate: Sentence with free variables.
"x+y = z."
"He was two metres tall."

Statement: Sentence which is unambiguously either true or false.
"For all real numbers x,y there is a real number z such that x+y = z."
"Samuel Marsden was two metres tall."

Examples:

"$x > 0$."
"For all $x \neq 0$, $x^2 > 0$."
"For all real numbers $x \neq 0$, $x^2 > 0$."

Predicates become statements by assigning values to or stating limits on the free variables (in which case they become "bound variables").

Quantifiers

Universal quantifier: For all $\forall$
Existential quantifier: There exists $\exists$

For all real numbers x, $x^2–1 = 0$. $\forall x \in R,\ x^2 - 1 = 0$.

For some real number x, $x^2–1 = 0$.
There exists a real number x such that $x^2–1 = 0$.
$\exists x \in R,\ x^2 - 1 = 0$.

Example
(p.9)
Suppose z is a free variable that refers to fish.

Give an example of a predicate A(z) such that the statement $\forall z,\ A(z)$ is true.

Give an example of a predicate B(z) such that the statement $\forall z,\ B(z)$ is false but $\exists z,\ B(z)$ is true.

Sentences can contain several quantifiers. Their order is important.

Let x and y refer to real numbers.
Predicate with two free variables: $y^2 = x$.
Statements:
$\forall x, \exists y\ \ y^2 = x$
$\exists x, \forall y\ \ y^2 = x$
$\forall y, \exists x\ \ y^2 = x$
and three more possible ways of quantifying this predicate.

Another example (p.5)
"For every poison there is a chemical that is the antidote."
"There is a chemical that is the antidote for every poison"

§2.2
Implication
(A, B are predicates or statements.)
"If A then B." "A implies B." $A \Rightarrow B$.
A = hypothesis, B = conclusion.

e.g. "If x is an integer then $x^2 \geq x$."
e.g. "If x is an integer then $x^3 \geq x$."
$\forall x,\ x \in Z \Rightarrow x^3 \geq x$.

e.g. Let U = {1, 2, 3}, X = {1}, Y = {1, 2}
Then X $\subseteq$ Y, i.e.
for all $x \in U$, $x \in X \Rightarrow x \in Y$.
Consider cases:

$x = 1:\ \ 1 \in X \Rightarrow 1 \in Y$

$x = 2:\ \ 2 \in X \Rightarrow 2 \in Y$

$x = 3:\ \ 3 \in X \Rightarrow 3 \in Y$

These three latter statements are all true, being special cases of the statement

"for each $x \in U$, $x \in X \Rightarrow x \in Y$."

In particular, note that conditional statements may be true in some cases even if the hypothesis or the conclusion is false. The truth of an implication is specified for each possible combination of truth values of its components by the following "truth table":

(p.12)

A	B	$A \Rightarrow B$
T	T	T
T	F	F
F	T	T
F	F	T

Compound statements and truth tables

(§2.4)

Basic connectives:

"not" $\sim$

"and" $\wedge$

"or" $\vee$

"implies" $\Rightarrow$

'"equivalent" $\Leftrightarrow$

Their definitions are in terms of truth tables.

A	B	$\sim A$	$A \wedge B$	$A \vee B$	$A \Rightarrow B$	$A \Leftrightarrow B$
T	T	F	T	T	T	T
T	F	F	F	T	F	F
F	T	T	F	T	T	F
F	F	T	F	F	T	T

Tautology: true for all values of the free variables in its components.

e.g. $A \vee \sim A$

Contradiction: false for all values of the free variables in its components.

e.g. $A \wedge \sim A$

Equivalence: P and Q are equivalent if P $\Leftrightarrow$ Q is a tautology.

e.g. if P is $A \wedge \sim B$ and Q is $\sim (A \Rightarrow B)$, then P and Q are equivalent.