The fundamental definition of **continuity** of a function $f: X \to Y$ where $X, Y \subseteq \mathbf{R}$ is:

$$\forall \varepsilon > 0 \ \exists \delta > 0: \ |x - y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon \qquad \textbf{[1]}$$

We can generalize this definition to any set X with a **distance function** $d: X \times X \to \mathbf{R}$ satisfying:
(i) $d(x,y) \geq 0$, $d(x.y) = 0 \Leftrightarrow x = y$ (ii) $d(x,y) = d(y,x)$ (iii) $d(x,z) \leq d(x,y) + d(y,z)$
The pair (X,d) is called a **metric space**.

Continuity can thus be defined between any pair of metric spaces (X, d_X) and (Y, d_Y):

$$\forall \varepsilon > 0 \ \exists \delta > 0: \ d_X(x,y) < \delta \Rightarrow d_Y(f(x), f(y)) < \varepsilon \qquad \textbf{[2]}$$

It is possible to frame this definition very neatly in terms of open sets a generalization of open intervals in **R**.

We define the **open ε-ball** $B_\varepsilon(x) = \{x \in X: d(x,y) < \varepsilon\}$. In **R** this is simply $B_\varepsilon(x) = \{x \in \mathbf{R}: |x - y| < \varepsilon\}$.
An **open set** $O \subseteq X$ is a set such that $\forall x \in O \ \exists B_\varepsilon(x) \subseteq O$.
It is straightforward to prove a set O is open if and only if it is a union of open balls.

We can rephrase our continuity definition: $\forall \varepsilon > 0 \ \exists \delta > 0: \ y \in B_\delta(x) \Rightarrow f(y) \in B_\varepsilon(f(x))$ **[3]**
i.e. $\forall B_\varepsilon(f(x)) \ \exists B_\delta(x): \ f(B_\delta(x)) \subseteq B_\varepsilon(f(x))$ **[4]**

An **open neighbourhood** (nhd) of $x \in X$ is any open set $x \in O \subseteq X$. We can then rephrase our definition of continuity in terms of open neighbourhoods:

Given any open nhd V of $f(x)$, there exists an open nhd U of x such that $f(U) \subseteq V$, **[5]**

We are then in a good position to define a continuous isomorphism between any two metric spaces:

$f: X \to Y$ is called a **homeomorphism** $\Leftrightarrow$ f is one to one and onto (i.e. a bijection)
and both f and f^{-1} are continuous.

Homeomorphism is the natural equivalence relation which preserves all the properties of continuity between different spaces, for example a hollow cube is homeomorphic to a sphere, but a torus or Klein bottle is not. However this definition does not coincide naturally with metric spaces and distance functions.

We say two metric spaces are **metrically equivalent** if there is a bijection $f: X \to Y$ such that $d_X(x,y) = d_Y(f(x), f(y)) \ \forall x, y \in X$.

It turns out that metric equivalence is not at all the same thing as homeomorphism. Homeomorphic metric spaces may not be metrically equivalent and so metric spaces are not really the natural objects to study continuity, although they arose axiomatically from the original definition of continuity in terms of distance.

Example: Thet two metric spaces (X, d_1) and (X, d_2) $X = (0,1]$ are not metrically equivalent :

$$d_1(x,y) = |x - y|, \quad d_2(x,y) = \left|\frac{1}{x} - \frac{1}{y}\right|$$

We can see this at once because many of the points in (X, d_2) are distance apart much greater than 1 while all pairs in (X, d_1) are closer then 1 apart. There can thus be no metric equivalence possible between them. However they are homeomorphic. In fact $y = \dfrac{1}{x}$ is a metric equivalence between (X, d_2) and $[1, \infty)$ with the standard metric $d_1(x,y) = |x - y|$ and this is also necessarily a homeomorphism. But $y = \dfrac{1}{x}$ is also a continuous bijection on $(0, \infty)$ using only the standard metric and is also its own continuous inverse so $[1, \infty)$ is also homeomorphic with (X, d_1). Hence (X, d_1) and (X, d_2) are homeomorphic.

To provide a distance-free definition of continuity, we resort to the definition of topological space:

A **topological space** is a pair (X,τ) where X is a set and τ is a collection of open subsets of X defined by:

$$\text{(i) } O_i \in \tau,\ \forall i \in I \Rightarrow \bigcup_{i\in I} O_i \in \tau \quad \text{(ii) } O_1,\ \ldots,O_n \in \tau \Rightarrow \bigcap_{i=1}^{n} O_i \in \tau$$

That is **arbitrary unions** and **finite intersections** of open sets are open.

This breaks the symmetry between union and intersection which characterizes set theory and Boolean logic in which we can exchange $\cup$ and $\cap$ (or $\vee$ and $\wedge$ in logic) to gain dual statements like De Morgan's laws.

Open sets such as (0,1) do not include their boundaries, while their complements, closed sets such as [0,1] (or $(-\infty,0]\cup[1,\infty)$ which is the closed complement of (0,1) in **R)** do. Consequently only finite intersections of open sets are open and only finite unions of closed sets are closed. For any space X, both X and $\varnothing$ are open since they are the null union and intersection of open sets. Consequently both are also closed.

Example: $\bigcap_{i=1}^{\infty}\left(-\frac{1}{n},1+\frac{1}{n}\right)=[0,1]$, but $\bigcup_{i=1}^{\infty}\left(\frac{1}{n},1-\frac{1}{n}\right)=(0,1)$.

The key transition between metric and topological spaces is to show that the definition of continuity carries across in a natural way to the following definition. $f:(X,\tau)\to(Y,\omega)$ between topological spaces is continuous iff: for every open set O in Y, $f^{-1}(O)$ is open in X. **[6]**

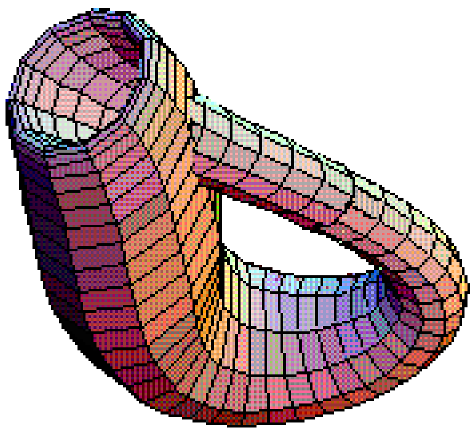
$$i.e.\ O\in\tau\Rightarrow f^{-1}(O)\in\omega.. \quad \textbf{[7]}$$

The following theorem completes the equivalence of this definition of continuity with all our previous definitions.

Theorem: $f:X\to Y$ is a continuous function of metric spaces if and only if for every open set O in Y, $f^{-1}(O)$ is open in X.

proof:
($\Rightarrow$) Suppose f is continuous and $O\subseteq Y$ open. If $f^{-1}(O)=\varnothing$ then it is open so assume the contrary. $x\in f^{-1}(O)\Rightarrow f(x)\in O$ open so $\exists B_\varepsilon(f(x))\subseteq O$. Hence by continuity $\exists B_\delta(x)$: $f(B_\delta(x))\subseteq B_\varepsilon(f(x))\subseteq O$. But then $B_\delta(x)\subseteq f^{-1}(O)$ so we have found an open ball around any $x\in f^{-1}(O)$ making it open.

($\Leftarrow$) Suppose O open $\Rightarrow f^{-1}(O)$ open. Consider $B_\varepsilon(f(x))$. Since this is open, $f^{-1}(B_\varepsilon(f(x)))$ is also open.and contains x. Hence there exists an open nhd of x in $f^{-1}(B_\varepsilon(f(x)))$ i.e. $\exists B_\delta(x)\subseteq f^{-1}(B_\varepsilon(f(x)))$. But this is the same thing as saying



The Klein bottle shown above and the sphere are not homeomorphic.
There is no continuously invertible bijection between them.