Infinite Sequences

Defn An infinite sequence of real numbers is a function whose domain is $\mathbb{N}$ (positive integers). We write it as $a_1, a_2, \ldots, a_n, \ldots$ or $\{a_n\}$. a_n is called the nth term and a sequence is often defined by giving a_n.

Examples $a_n = 1/n$, $\quad 1,\ \frac{1}{2},\ \frac{1}{3}, \ldots$

$a_{2n-1} = 1, a_{2n} = 2n^2 \quad 1, 2, 1, 8, 1, 18, \ldots$

$a_1 = a_2 = 1, \quad a_{n+1} = a_n + a_{n-1}\ (n \geq 2)$ (Recursion formula)

The last example gives the Fibonacci numbers $1, 1, 2, 3, 5, \ldots$.

We can graph a sequence in the plane or on a line. Note that different terms of a sequence may have the same value. Our main concern is what happens to the terms a_n of the sequence as n becomes large. This behaviour will be unaffected by any alteration to a *finite* number of terms.

Defn A sequence $\{a_n\}$ *converges to a limit* l if given any $\epsilon > 0$ we can find an N (usually depending on ϵ) such that

$$|a_n - l| < \epsilon \quad \text{for all} \quad n > N.$$

We write $\lim_{n\to\infty} a_n = l$ or $a_n \to l$ as $n \to \infty$. A sequence which does not converge ins called *divergent*. (This includes the possibility of finite or infinite oscillations, e.g. $(-1)^n$.)

Example $\{\frac{1}{n}\}$. Given $\epsilon > 0$ there exists N such that $\frac{1}{N} < \epsilon$. Then for $n > N$ we have $\frac{1}{n} < \frac{1}{N} < \epsilon$ and hence $|\frac{1}{n} - 0| < \epsilon$ for all $n > N$. So $\frac{1}{n} \to 0$ as $n \to \infty$.

Direct application of the definition is often inconvenient so we need results about combinations of limits.

Lemma If $a_n \to l$ as $n \to \infty$ then $\{a_n\}$ is bounded, i.e. there exists $K > 0$ such that $|a_n| \leq K$ for all n.

Proof Given $\epsilon = 1$ there exists N such that $|a_n - l| < 1$ for $n > N$. So for $n > N$
$|a_n| = |a_n - l + l| \leq |a_n - l| + |l| < |l| + 1$. Take $K = \max\{|a_1|, \ldots, |a_N|, |L| + 1\}$. Then $|a_n| \leq K$ for all n.

We can't simply take $K = |l| + 1$ because a bigger value of $|a_n|$ might occur among the first N terms.

Defn If $a_n \to 0$ as $n \to \infty$ then $\{a_n\}$ is called a *null* sequence.

Theorem If $\{a_n\}$ and $\{b_n\}$ are null sequences then so is $\{a_n + b_n\}$.

Proof Given $\epsilon > 0$ there exist N_1, N_2 such that $|a_n| < \epsilon/2$ i.e. $-\epsilon/2 < a_n < \epsilon/2$ for all $n > N_1$ and $-\epsilon/2 < b_n < \epsilon/2$ for all $n > N_2$. Then for $n > N = \max(N_1, N_2)$ both inequalities hold so adding gives $-\epsilon < a_n + b_n < \epsilon$ or $|a_n + b_n| < \epsilon$. So $\{a_n + b_n\}$ is also a null sequence.

Theorem If $\{a_n\}$ is a null sequence and $\{b_n\}$ is a bounded sequence then $\{a_nb_n\}$ is a null sequence.

Proof There exists a $K > 0$ such that $|b_n| < K$ for all n. Also given $\epsilon > 0$ there exists an N such that $|a_n| < \frac{\epsilon}{K}$ for all $n > N$. So $|a_nb_n| < K \cdot \frac{\epsilon}{K} = \epsilon$ for all $n > N$.
i.e. $\{a_nb_n\}$ is a null sequence.

Cor If $\{a_n\}$ is a null sequence and c is a constant, then $\{ca_n\}$ is a null sequence.

Theorem If $a_n \to l, b_n \to m$ as $n \to \infty$, then
(i) $a_n + b_n \to l + m$
(ii) $a_nb_n \to lm$ (all as $n \to \infty$)

Proof (i) $\{a_n - l\}$ and $\{b_n - m\}$ are null sequences.
Therefore so is their sum $\{a_n + b_n - (l + m)\}$.

(ii) $a_nb_n - lm = (a_n - l)b_n + l(b_n - m)$. In the first term on the RHS $\{a_n - l\}$ is a null sequence and $\{b_n\}$ is a bounded sequence. so their product is a null sequence. The second term is the null sequence $\{b_n - m\}$ multiplied by the constant l and so is also a null sequence. So the RHS is the sum of two null sequences and hence a null sequence.
i.e. $a_nb_n \to lm$ as $n \to \infty$.

Theorem If $a_n \to l$ and $b_n \to m$ as $n \to \infty$, and $m \neq 0$, then $\dfrac{a_n}{b_n} \to \dfrac{l}{m}$ as $n \to \infty$.

Proof It is enough to show that $\dfrac{1}{b_n} \to \dfrac{1}{m}$. (Why?)

i.e. that $\dfrac{1}{b_n} \to \dfrac{1}{m} = \dfrac{m - b_n}{b_nm}$ is a null sequence.

Now we can choose N such that for all $n > N$ we have $|b_n| > \frac{1}{2}|m|$, and so $\dfrac{1}{|b_nm|} < \dfrac{2}{|m|^2}$.

Then $\{b_n - m\}$ is a null sequence and $\left\{\dfrac{1}{b_nm}\right\}$ is a bounded sequence. Hence by an earlier result $\left\{\dfrac{m - b_n}{b_nm}\right\}$ is a null sequence.

NB To see that $|b_n| > \frac{1}{2}|m|$ we note that $\big||b_n| - |m|\big| \leq |b_n - m| < \epsilon = \frac{1}{2}|m|$ for $n >$ (suitable) N.
But then $-\frac{1}{2}|m| < |b_n| - |m| < \frac{1}{2}|m|$ i.e. $\frac{1}{2}|m| < |b_n| < \frac{3}{2}|m|$.

Theorem A sequence can have at most one limit.

Sandwich Theorem Suppose that $a_n \to l$ and $b_n \to l$ as $n \to \infty$. If $a_n \leq x_n \leq b_n$ $(n = 1, 2, \dots)$, then $x_n \to l$ as $n \to \infty$.

Proof Given $\epsilon > 0$ there exist N_1, N_2 such that

$$|a_n - l| < \epsilon \quad \text{i.e.} \quad l - \epsilon < a_n < l + \epsilon \quad \text{for } n > N_1$$
$$|b_n - l| < \epsilon \quad \text{i.e.} \quad l - \epsilon < b_n < l + \epsilon \quad \text{for } n > N_2.$$

Then for $n > \max\{N_1, N_2\}$ we have $l - \epsilon < a_n \leq x_n \leq b_n < l + \epsilon$ i.e. $l - \epsilon < x_n < l + \epsilon$ or $|x_n - l| < \epsilon$ for $n > N$.

Monotone Sequences

A sequence $\{a_n\}$ is *increasing* ($\nearrow$) if $a_n \leq a_{n+1}$ for all $n \geq 1$. Similarly for a *decreasing* sequence. A sequence is called *monotonic* if it is either $\nearrow$ or $\searrow$.

Theorem A monotonic sequence coverges if and only if it is bounded.

Proof Clearly an unbounded sequence does not converge. Suppose $a_n \nearrow$ and let $L = \text{lub}\{a_n\}$. We know L exists since $\{a_n\}$ is bounded. If $\epsilon > 0$ then there exists N such that $a_N > L - \epsilon$ (by the defn of lub). But then if $n > N$ we have $a_n > a_N$ since $a_n \nearrow$.
Hence $L - \epsilon \leq a_n \leq L$ or $0 \leq L - a_n \leq \epsilon$ for all $n \geq N$, i.e. $a_n \to l$ as $n \to \infty$.

Similarly if $a_n \searrow$ (and is bounded) then $a_n \to \text{glb}\{a_n\}$.

NB As we have already noted it may not be easy to decide whether a given increasing sequence is bounded above. For example what happens to the sequence

$$1, \quad 1 + \frac{1}{2}, \quad 1 + \frac{1}{2} + \frac{1}{3}, \quad 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}, \quad \ldots ?$$

Defn A *subsequence* of a sequence $\{a_n\}$ is a sequence of the form

$$a_{n_1}, \quad a_{n_2}, \quad a_{n_3}, \quad \ldots$$

where $n_i \in \mathbb{N}$ and $n_1 < n_2 < n_3 < \cdots$.

Lemma Any sequence $\{a_n\}$ contains a subsequence which is either increasing or decreasing.

Proof Call n a "peak" point of the sequence $\{a_n\}$ if $a_m < a_n$ for all $m > n$.

Case 1 (The sequence has infinitely many peak points.)
In this case if $n_1 < n_2 < n_3 < \cdots$ are the peak points, then $a_{n_1} > a_{n_2} > a_{n_3} > \cdots$ and so $\{a_{n_k}\}$ is the desired ($\searrow$) sequence.

Case 2 (Only finitely many peak points.)
Let n_1 be greater than all the peak points. then since n_1 is not a peak point there is an $n_2 > n_1$ such that $a_{n_2} \geq a_{n_1}$. Since n_2 is also not a peak point there is an $n_3 > n_2$ such that $a_{n_3} \geq a_{n_2}$. continuing in this way we get the desired ($\nearrow$) sequence.

Cor Every bounded sequence has a convergent subsequence.

Remark In general there may be many such subsequences. We now establish a necessary and sufficient condition for convergence which plays a fundamental role in more advanced work.

Defn A sequence $\{a_n\}$ is a *Cauchy* sequence if for every $\epsilon > 0$ there exists and N such that $|a_n - a_m| < \epsilon$ whenever $m, n > N$, i.e. $\lim_{m,n\to\infty} |a_m - a_n| = 0$.

Theorem A sequence $\{a_n\}$ in $\mathbb{R}$ converges if and only if it is a Cauchy sequence.

Proof It is easy to show that a convergent sequence is Cauchy.

For the converse we first show that a Cauchy sequence $\{a_n\}$ is bounded. Take $\epsilon = 1$ in the definition. Then there exists N such that $|a_m - a_n| < 1$ for all $m, n > N$. In particular $|a_m| = |a_m - a_{N+1} + a_{N+1}| < |a_m - a_{N+1}| + |a_{N+1}| < 1 + |a_{N+1}|$ (since $N + 1 > N$).
So if $K = \max\{|a_1|, |a_2|, \ldots , |a_N|, |a_{N+1} + 1|\}$ then $|a_m| \leq K$ for all m, i.e. the sequence is bounded and hence has a convergent subsequence.

The final step (which we omit) is to show that if a subsequence $\{a_{n_k}\}$ converges to l then the original Cauchy sequence must also converge to l.

Remark Convergence of sequences can be defined in any metric space. A metric space in which every Cauchy sequence converges is called *complete*. The last result shows that $\mathbb{R}$ with the usual absolute value metric *is* complete. This definition of completeness is equivalent to the earlier one in terms of the least upper bound property. In fact, as previously noted, a standard construction of the real numbers is to define them as certain equivalence classes of Cauchy sequences of rational numbers.