If $f: A \to B$ is a function, and there exists a function $g: B \to A$ satisfying $g \circ f(x) = x$ for all $x \in A$ and $f \circ g(y) = y$ for all $y \in B$, then f is one-to -one and onto, and $g = f^{-1}$, the inverse function of f.

(that is, $Ran(f) = \{f(a):\ a \in A\}$) then
$f: A \to Ran(f)$ is a bijection.

Composition

If $f: A \to B$ and $g: B \to C$ are functions, then the *composition* of f and g is the function

$$g \circ f: A \to C$$

defined by

$$g \circ f(a) = g(f(a)).$$

- Composition is associative.(Prop 5.2.2)

- Composition of one-to-one functions is one-to-one.(Thm 5.2.3.1)

- Composition of onto functions is onto. (Thm 5.2.3.2)

We return briefly to relations in general. If ρ is a relation from A to B, then the *inverse relation* of ρ is the relation from B to A defined by

$$\rho^{-1} = \{(b,a) \in B \times A\ :\ (a,b) \in \rho\}.$$

Theorem (variation of Th 5.2.5) Let $f: A \to B$ be a function. Then the inverse relation f^{-1} from B to A is a function if and only if f is one-to-one and onto.

In this case, f^{-1} is called the *inverse function* of *f*.

It satisfies $f^{-1} \circ f(x) = x$ for all $x \in A$ and
$f \circ f^{-1}(y) = y$ for all $y \in B$.

The converse is also true:

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Chapter 5. Functions

A **function** (or **mapping**) $f: A \to B$ is a relation from A to B satisfying

$\forall a \in A \ \exists b \in B \ (a,b) \in f$
(i.e. domain $f = A$)

$\forall a \in A \ \forall b_1, b_2 \in B \ [(a,b_1) \in f \wedge (a,b_2) \in f] \Rightarrow b_1 = b_2$
(i.e, f is a "single-valued" relation.)

In other words, for each $a \in A$ there is one and only one $b \in B$ such that $(a,b) \in f$.

If f is a function, we write
$f(a) = b$ to mean that $(a,b) \in f$.

A function $f: A \to B$ is

- **one-to-one** if
$\forall a_1, a_2 \in A \ \forall b \in B$ if $(a_1, b) \in f$ and $(a_2, b) \in f$ then $a_1 = a_2$

In other words, if $f(a_1) = f(a_2)$ then $a_1 = a_2$, (this is Prop 5.1.10) or "distinct elements of A have distinct images under *f*."

- **onto** if $\forall b \in B \ \exists a \in A \ (a,b) \in f$.

In other words, "every element of B is the image of some element of A under *f*."

- a **bijection** if it is one-to-one and onto.