THE UNIVERSITY OF AUCKLAND

SECOND SEMESTER, 2001
Campus: City

MATHEMATICS

Principles of Mathematics

(Time allowed: THREE hours)

NOTE: Attempt **ALL EIGHT** questions. Each question is worth 20 marks.
This is an open book examination, and the use of calculators is permitted.

1. (a) Prove that if x is an integer such that x^2 is odd, then x is odd. [8 marks]

(b) Let (a_n) be a sequence of real numbers, and let $L \in \mathbf{R}$. We say that $\lim_{n\to\infty} a_n = L$ if

$$(\forall \varepsilon \in \mathbf{R}^+)(\exists N \in \mathbf{N})(\forall n \in \mathbf{N}) \text{ if } n > N \text{ then } |a_n - L| < \varepsilon.$$

Write down the negation of this condition, i.e write down in similar quantified form what it means when we say it is *not* the case that $\lim_{n\to\infty} a_n = L$. [4 marks]

(c) Two predicates are *consistent* if it is possible for both of them to be true simultaneously. For example :"x is an even integer" is consistent with "x is a multiple of 3". Two predicates are *independent* if it is possible for them both to be true simultaneously, it is possible for each to be true while the other is false, and it is possible for both to be false simultaneously. Let P be the predicate $p \wedge \sim (q \vee r)$ and let Q be the predicate $p \wedge (\sim q \vee \sim r)$.

(i) Determine whether or not P,Q are consistent. [4 marks]

(ii) Determine whether or not P,Q are independent. [4 marks]

2. (a) Suppose that X,Y are sets and that $f: X \to Y$ is a function. A relation ρ is defined on X by $a\rho b$ if and only if $f(a) = f(b)$.

(i) Show that ρ is an equivalence relation on X. [4 marks]

(ii) Suppose further that $X = Y = \mathbf{R}$ and $f: X \to Y$ is defined by

$$f(x) = \begin{cases} \dfrac{x}{|x|} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}.$$

Describe the equivalence classes determined by ρ. [4 marks]

(b) Let $A = \{1,2,3,4,5\}$, and let ρ be the following relation on A:

$$\rho = \{(1,1),(2,2),(3,3),(4,4),(5,5),(3,1),(4,1),(4,2),(4,3),(5,1),(5,3)\}.$$

(i) Show that ρ is antisymmetric. [3 marks]

ρ is a partial ordering on A. (You need not prove this.)

(ii) Show that ρ is not a total ordering on A. [3 marks]

(iii) Find a totally ordered subset of A with 3 elements in it. [3 marks]

(iv) Find a non-empty subset of A which has no upper bound in A. [3 marks]

3. Find and prove a formula for the number of diagonals of a convex polygon with n sides where $n \geq 3$. [The polygon is not assumed to be regular. *Convex* means that a line segment joining any two vertices lies within or on the polygon. A *diagonal* is a line segment joining two non-adjacent vertices.] [20 marks]

4. (a) Prove that for all $a, b \in \mathbf{Z}$ and for all $m \in \mathbf{N}$,

if $a \equiv b \bmod m$, then $\gcd(a,m) = \gcd(b,m)$. [10 marks]

(b) Find a greatest common divisor $d(x)$ and least common multiple $m(x)$ of the polynomials

$$a(x) = x^3 + x + \bar{2}, \quad b(x) = x^3 + x^2 + \bar{2}x + \bar{2} \quad \text{in } \mathbf{Z}_3[x].$$

[You must use arithmetic operations appropriate to $\mathbf{Z}_3$ in your calculations.] [10 marks]

5. (a) Let $(\mathbf{Z}_n, +_n)$ be the group of integers modulo n under addition, and $(U(n), \cdot_n)$ be the group of invertible elements of $\mathbf{Z}_n$ under multiplication.

(i) Write out the Cayley tables for $\mathbf{Z}_4$ and $U(8)$. [5 marks]

(ii) Show that $\mathbf{Z}_4$ and $U(8)$ are *not* isomorphic. [5 marks]

(b) Show that the identity element of a group is unique. [5 marks]

(c) Let $(G, *)$ be a group, not necessarily commutative, and let $a, b \in G$ such that $b^{-1} * a * b = a^2$. Show that for all $s \in \mathbf{N}$, $b^{-1} * a^s * b = a^{2s}$. [5 marks]

6. (a) Prove each of the following properties of $\mathbf{R}$ using the field axioms:

(i) Given $a, b \in \mathbf{R}$, with $a \neq 0$ there is exactly one $x \in \mathbf{R}$ such that $ax = b$. [4 marks]

(ii) If $ab = 0$ then $a = 0$ or $b = 0$. [4 marks]

(b) Using the order axioms prove that

for all x,y,z in $\mathbf{R}$, if $x < y$ and $z < 0$ then $xz > yz$. [4 marks]

(c) Use the concept of the least upper bound and the completeness axiom for $\mathbf{R}$ to show that there is a real number x such that $x^3 = 3$. [8 marks]

7. (a) Consider the sequence (u_n) where $u_n = \left(1 - \frac{1}{n}\right)\sin^2\left(\frac{n\pi}{2}\right)$, $n = 1, 2, 3, \cdots$.

(i) Determine whether or not (u_n) is monotone. [3 marks]

(ii) Determine whether or not (u_n) is bounded. [3 marks]

(iii) Determine whether or not (u_n) has distinct terms. [3 marks]

(iv) Find the greatest lower bound and least upper bound of (u_n) and determine whether or not either is an element of (u_n). [3 marks]

(b) Prove from first principles that any convergent sequence is a Cauchy sequence. [8 marks]

8. (a) Suppose f, g are continuous functions on $[a, b]$. Show from first principles that $f - g$ is continuous on $[a, b]$. [10 marks]

(b) Prove the Cauchy Mean Value Theorem: If the real functions f and g are differentiable and $g'(x) \neq 0$ on $[a, b]$, then there exists $c \in (a, b)$ such that

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}.$$

[Hints: Consider $F(x) = f(x) + p.g(x)$ and determine a constant p for which $F(a) = F(b)$. Now apply Rolle's theorem.] [10 marks]