DEPARTMENT OF MATHEMATICS

MATHS 255FC	Assignment 8	Due 26th September 2001

1. Prove each of the following from the axioms given in the handout on real numbers.

(a) Given $a, b \in \mathbf{R}$, show there is a unique x such that $a + x = b$.
(b) If $x = b - a$ is defined to be this x, and $-a$ is defined to be $0 - a$ show:
(i) $b - a = b + (-a)$ (ii) $a(b - c) = ab - ac$ (iii) $0.a = a.0 = 0$ (iv) $ab = ac,\ a \neq 0 \Rightarrow b = c$
(c) (i) $x < y \Rightarrow x + z < y + z$ (ii) $1 > 0$ (iii) $x < y \Rightarrow -x > -y$
(d) $A, B \subseteq \mathbf{R}$, $A, B \neq \varnothing$, $A \subseteq B$ and B is bounded above, show $\text{lubA} \leq \text{lubB}$.

2. (a) Show from the definition of absolute value in the real numbers handout that $|x + y| \leq |x| + |y|$.

(b) Use (a) to prove by induction that $|x_1 + x_2 + \ \dots\ + x_n| \leq |x_1| + |x_2| + \ \dots\ + |x_n|$.

(c) Show that the distance function $d(x, y) = |x - y|$ obeys the triangle law: $d(x, z) \leq d(x, y) + d(y, z)$.

3. Find the least upper bound and greatest lower bound of each of the following subsets of **R** if they exist and determine if either of these is an element of the set concerned.

(i) $\mathbf{R} \setminus \{0\}$ (ii) $(-\infty, -1)$ (iii) $[-\pi, 2)$ (iv) $[-\pi, 2) \cap (\mathbf{R} \setminus \mathbf{Q})$ (v) $\left\{\frac{1}{n^2} :\ n \in \mathbf{N}\right\}$.

4. For each of the following sequences, determine whether or not it is:

(a) convergent and if so find its limit,
(b) bounded and if so find a convergent subsequence
(c) find a subsequence which is increasing. or one which is decreasing, or both if possible.

(i) $\{1^n + (-1)^n :\ n \in \mathbf{N}\}$ (ii) $\left\{\frac{1}{n^2} :\ n \in \mathbf{N}\right\}$ (iii) $\left\{\frac{1^n}{n^2} + \frac{(-1)^n}{n^2} :\ n \in \mathbf{N}\right\}$ (iv) $\{n! :\ n \in \mathbf{N}\}$ (v) $\left\{\frac{n!}{n^n} :\ n \in \mathbf{N}\right\}$

Prove each of the following specifically from the axioms given in the handout on real numbers.

(a) Given $a, b \in \mathbf{R}$, show there is a unique x such that $ax = b$.
There exists $y : ay = 1$ (multiplicative inverse). Let $x = by$ then $ax = aby = bay = b.1 = b$.
If $ax = b,\ ax' = b$ then $x - x' = 1(x - x') = ay(x - x') = ya(x - x') = yax - yax' = yb - yb = 0$, so $x = x'$.

If $x=b/a$ is defined to be $ax = b$, show:
(i) $a/b + c/d = (ad + bc)/bd$ if $b, d \neq 0$. (ii) $(a/b).(c/d) = ac/bd$ if $b, d \neq 0$.
(i) Let $a = x.b,\ c = y.d$, then $a/b + c/d = x + y$.
Now consider $z = (ad + bc)/bd = (xbd + byd)/bd = (x + y)bd/bd$, then $z(bd) = (x + y)bd$ so $z = x + y$.
(ii) $z = ac/bd = xbyd/bd = xybd/bd$, so $zbd = xybd$, and $z = xy$.

(b) (i) $x < y,\ y < z \Rightarrow x < z$ (ii) $x < y,\ z > 0 \Rightarrow xz < yz$.
(i) $x < y,\ y < z \Leftrightarrow y - x,\ z - y \in P \Rightarrow z - y + y - x = z - x \in P \Leftrightarrow x < z$.
(ii) $x < y,\ z > 0 \Leftrightarrow y - x,\ z \in P \Rightarrow z(y - x) = zy - zx \in P \Rightarrow xz < yz$.

(c) (i) $|xy| = |x|.|y|$ (ii) $||x| - |y|| \leq |x - y|$.
(i) $x.y \geq 0,\ |xy| = x.y = |x|.|y| = |-x|.|-y|,\ x.y < 0,\ |xy| = -x.y = |x|.|-y| = |-x|.|y|$.
(ii) $x, y \geq 0\ ||x| - |y|| = |x - y|,\ x, y < 0\ ||x| - |y|| = |-x - -y| = |x - y|$
$x \geq 0, y < 0\ ||x| - |y|| = |x + y| < |x + -y| = |x - y|$ since $x, -y$ both have the same sign but x, y have opposite sign. $x < 0, y \geq 0\ ||x| - |y|| = |-x + y| < |x - y|$ for the same reason.