THE UNIVERSITY OF AUCKLAND

FIRST SEMESTER, 2001
Campus: City

MATHEMATICS

Principles of Mathematics

(Time allowed: THREE hours)

NOTE: Answer ALL the questions. All questions carry equal marks.

1. Let $f : \mathbb{N} \to \mathbb{N}$ be given by $f(x) = x^3 + 5x$.

(a) Use a **direct proof** to show that if $n < k$ then $f(n) < f(k)$.

(b) Use a **proof by contraposition** to show that if $f(n) < f(k)$ then $n < k$.

(c) Use a **proof by contradiction** to show that if $f(n) = f(k)$ then $n = k$.

(d) Use a **proof by cases** to prove that $f(n)$ is a multiple of 3 for all $n \in \mathbb{N}$. [You may assume that every integer can be written uniquely in the form $3k$, $3k + 1$ or $3k + 2$.]

2. (a) We define the relation ρ on $\mathbb{Z}$ by declaring that, for $x, y \in \mathbb{R}$,

$$x \,\rho\, y \qquad \Longleftrightarrow \qquad x \leq y^2.$$

Which of the following are true and which are false? Give brief reasons for you answers.

(i) ρ is reflexive.

(ii) ρ is symmetric.

(iii) ρ is antisymmetric.

(iv) ρ is transitive.

(b) Let ρ and σ be equivalence relations on a set A. Define a relation τ on A by declaring that, for $a, b \in A$, $a \,\tau\, b$ if and only if $(a \,\rho\, b \wedge a \,\sigma\, b)$. Show that τ is an equivalence relation.

3. A *blob* function is a function $f : \mathbb{R} \to \mathbb{R}$ such that if $x, y \in \mathbb{R}$ with $|x - y| < 1$ then $|f(x) - f(y)| < 1$.

(a) (i) Give an example of a blob function which is not continuous

(ii) Give an example of a continuous function $f : \mathbb{R} \to \mathbb{R}$ which is not a blob function.

(b) Show that if f and g are blob functions then $f \circ g$ is a blob function.

(c) Show that if f and g are blob functions then the function $h : \mathbb{R} \to \mathbb{R}$ defined by

$$h(x) = \frac{f(x) + g(x)}{2}$$

is a blob function.

4. (a) Let $a, b \in \mathbb{N}$. Let $l \in \mathbb{N}$ be a common multiple of a and b. Prove that the following are equivalent:

I. for every positive common multiple m of a and b, $l \leq m$.

II. for every common multiple m of a and b, $l \mid m$.

[Hint: if m is a common multiple of a and b, and $m = ql + r$, you can show that r is a common multiple of a and b.]

(b) Let $a, b \in \mathbb{N}$. Suppose that there exist $x, y \in \mathbb{Z}$ with $ax + by = 1$. Show that a and b are relatively prime.

5. (a) Let $(\mathbb{Z}_n, +_n)$ be the group of integers modulo n under addition.

(i) Write out the Cayley Tables for $\mathbb{Z}_4$ and $\mathbb{Z}_2 \times \mathbb{Z}_2$ and show they are not isomorphic.

(ii) Show that $\mathbb{Z}_4$ is isomorphic to a geometrical symmetry group, demonstrating the isomorphism by using Cayley tables.

(b) (i) Show that in any group the inverse of an element is unique.

(ii) Prove that a group is commutative if and only if the function which maps every element to its inverse is an isomorphism.

6. (a) (i) Give an example of a number system which obeys the field and order axioms, but not the completeness axiom. Explain why your example does not satisfy the completeness axiom.

(ii) Give an example of a number system which obeys the field axioms but not the order axioms. Explain why your example does not satisfy the order axioms.

(b) Use the order axioms to prove:

(i) For arbitrary $x, y \in \mathbb{R}$ exactly one of the following holds: $x < y$, $y < x$, $x = y$.

(ii) For arbitrary $x, y \in \mathbb{R}$, $|x + y| \leq |x| + |y|$.

7. (a) Consider the sequence $\{u_n\}_{n=0}^{\infty}$, where $u_n = \dfrac{2}{1 + \mathrm{e}^{-n}}$.

(i) Show that $\{u_n\}_{n=0}^{\infty}$ is monotonic and bounded.

(ii) Find the greatest lower bound and least upper bound of $\{u_n\}_{n=0}^{\infty}$ and determine whether either is an element of $\{u_n\}_{n=0}^{\infty}$.

(iii) Using the first principles definition of a convergent sequence show that the sequence $\{u_n\}_{n=0}^{\infty}$ is convergent to its least upper bound.

(b) Suppose f is continuous on $[a, b]$ and $f(a) < k < f(b)$. Show from first principles that there is some $c \in [a, b]$ such that $f(c) = k$.

8. (a) Let f be a real valued function. Show that if f is continuous then $|f|$ is continuous.

(b) Let f be a continuous real valued function on $[a, b]$. Suppose that for each $x \in [a, b]$ there exists a $y \in [a, b]$ such that $|f(y)| \leq \frac{1}{2}|f(x)|$. Prove that there exists a $z \in [a, b]$ for which $f(z) = 0$. [Hint: either use the Extreme Value Theorem, or use a similar method to the proof of that theorem]