1. (a) The contrapositive of $A(n)$ is "If $n+2$ is not prime then n is not prime."

(b) The converse of $A(n)$ is "If $n+2$ is prime then n is prime."

(c) The negation of $A(n)$ is "n is prime but $n+2$ is not prime."

(d) $A(n)$ true for some $n \in \mathbb{N}$: for example, $A(5)$ is true.

(e) $A(n)$ is true for every n: for example, $A(7)$ is false.

(f) Since the contrapositive of $A(n)$ is equivalent to $A(n)$ itself, from (d) and (e) we see that the contrapositive is true for some $n \in \mathbb{N}$ but not for all $n \in \mathbb{N}$.

(g) The converse of $A(n)$ is true for some but not all n: for example, the converse of $A(7)$ is true but the converse of $A(9)$ is false.

2. (a) **Reflexive:** Let $x \in X$. Then $f(x) = f(x)$, so $f(x) \leq f(x)$, so $x \,\rho\, x$. Hence ρ is reflexive.

Transitive: Let $x, y, z \in X$ with $x \,\rho\, y$ and $y \,\rho\, z$. Then $f(x) \leq f(y)$ and $f(y) \leq f(z)$, so $f(x) \leq f(z)$, so $x \,\rho\, z$. Hence ρ is transitive.

(b) Suppose first that ρ is antisymmetric. Let $x, y \in X$ with $f(x) = f(y)$. Then $f(x) \leq f(y)$, so $x \,\rho\, y$, and $f(y) \leq f(x)$ so $y \,\rho\, x$. Since ρ is antisymmetric and $x \,\rho\, y$ and $y \,\rho\, x$, we have $x = y$. Thus f is one-to-one.

Conversely, suppose that f is one-to-one. Let $x, y \in X$ with $x \,\rho\, y$ and $y \,\rho\, x$. Then $f(x) \leq f(y)$ and $f(y) \leq f(x)$, so $f(x) = f(y)$, so (since f is one-to-one) $x = y$. Thus ρ is antisymmetric.

3. (a) Suppose that f is a blah function. We have $0+0=0$, so $f(0+0) = f(0)$, so $f(0) \cdot f(0) = f(x)$. But the only solutions of $x \cdot x = x$ in $\mathbb{R}$ are $x = 0$ and $x = 1$, and we cannot have $f(0) = 0$ (since $f : \mathbb{Z} \to \mathbb{R} \setminus \{0\}$). So $f(0) = 1$.

(b) Suppose that f is a blah function. Let $x \in \mathbb{Z}$. Then $f(x+(-x)) = f(0)$, so $f(x) \cdot f(-1) = f(0) = 1$ (by part (a)). Dividing by $f(x)$ (which we can do since $f(x) \neq 0$) we have $f(-x) = \dfrac{1}{f(x)}$.

(c) Let f and g be blah functions with $f(1) = g(1)$. For $n \in \mathbb{N}$ let P_n be the statement that $f(n) = g(n)$. We prove by induction that P_n is true for all n.

Base: We are given that $f(1) = g(1)$, so P_1 is true.

Inductive Step: Suppose $n \in \mathbb{N}$ with P_n true. Then $f(n) = g(n)$, so

$$\begin{aligned} f(n+1) &= f(n) \cdot f(1) && \text{since } f \text{ is blah} \\ &= g(n) \cdot f(1) && \text{by inductive hypothesis, and the fact that } f(1) = g(1) \\ &= g(n+1) && \text{since } g \text{ is blah.} \end{aligned}$$

Hence P_{n+1} is true.

Hence, by induction, P_n is true for all $n \in \mathbb{N}$.

(d) Let f and g be blah functions with $f(1) = g(1)$. To show that $f = g$ we must show that f and g have the same domain and that $f(x) = g(x)$ for all $x \in \operatorname{dom}(f)$. Well, $\operatorname{dom}(f) = \operatorname{dom}(g) = \mathbb{Z}$, so we only need to show that $f(x) = g(x)$ for all $x \in \mathbb{Z}$. So let $x \in \mathbb{Z}$. We have three cases to consider: $x > 0$, $x = 0$ and $x < 0$.

Case 1: When $x > 0$ we have $x \in \mathbb{N}$ so by (c) above we have $f(x) = g(x)$.
Case 2: When $x = 0$ we have $f(x) = 1 = g(x)$ by (a).
Case 3: When $x < 0$ we have $-x \in \mathbb{N}$, so $f(-x) = g(-x)$ by Case 1, so $\dfrac{1}{f(x)} = \dfrac{1}{g(x)}$ by (b), so $f(x) = g(x)$.

Hence in any case we have $f(x) = g(x)$, as required.

4. (a) We must show that the equation has a solution, and that the solution is unique. Note that, if we multiply the equation by $\overline{c}$ we get $\overline{c} \cdot_n \overline{a} \cdot_n \overline{x} = \overline{c} \cdot_n \overline{b}$, in other words $\overline{x} = \overline{c} \cdot_n \overline{b}$ (since $\overline{c} \cdot_n \overline{a} = \overline{ca} = \overline{1}$. This suggests our solution should be $\overline{x} = \overline{c} \cdot_n \overline{b}$. So we substitute this value into the equation:

$$\overline{a} \cdot_n (\overline{c} \cdot_n \overline{b}) = (\overline{a} \cdot_n \overline{c}) \cdot_n \overline{b} = \overline{1} \cdot_n \overline{b} = \overline{b},$$

so this is indeed a solution.

Now we must show that it is unique, so suppose that $\overline{x}$ and $\overline{y}$ are both solutions, in other words that $\overline{a} \cdot_n \overline{x} = \overline{b}$ and $\overline{x} \cdot_n \overline{y} = \overline{b}$. Then $\overline{a} \cdot_n \overline{x} = \overline{a} \cdot_n \overline{y}$, so $\overline{c} \cdot_n \overline{a} \cdot_n \overline{x} = \overline{c} \cdot_n \overline{a} \cdot_n \overline{y}$, i.e. $\overline{1} \cdot_n \overline{x} = \overline{1} \cdot_n \overline{y}$, so $\overline{1 \cdot_n x} = \overline{1 \cdot_n y}$, i.e. $\overline{x} = \overline{y}$, as required.

(b) We use Euclid's Algorithm:

35	1	0
16	0	1
3	1	−2
1	−5	11
0	16	−35

From this we see that $35 \cdot (-5) + 16 \cdot 11 = 1$. Multiplying by 3 we have $35 \cdot (-15) + 16 \cdot 33 = 3$, so $x = -15$, $y = 33$ is a solution, and the general solution is

$$x = -15 + 16t, y = 33 - 35t \qquad \text{for } t \in \mathbb{Z}.$$