DEPARTMENT OF MATHEMATICS

MATHS 255FC	Assignment 8 solutions	Due 9th May 2001

1. (a) Show first principles that the sequence $\left\{\frac{4n+1}{n+4},\ n=0,\ 1,\ 2,\ \ldots\right\}$ is convergent.

$$\left|\frac{4n+1}{n+4}-4\right|<\varepsilon \Leftrightarrow \left|\frac{4n+1-4n-16}{n+4}\right|<\varepsilon \Leftrightarrow \left|\frac{-15}{n+4}\right|<\varepsilon \Leftrightarrow 15<\varepsilon(n+4) \Leftrightarrow 15<\varepsilon n+4\varepsilon \Leftrightarrow 15-4\varepsilon<\varepsilon n$$

$\Leftrightarrow \frac{15-4\varepsilon}{\varepsilon}<n$ Hence choose $N(\varepsilon)=\frac{15-4\varepsilon}{\varepsilon}$ then $\forall\varepsilon>0,\exists N(\varepsilon)=\frac{15-4\varepsilon}{\varepsilon}>0:n>N\Rightarrow\left|\frac{4n+1}{n+4}-4\right|<\varepsilon$

[4 marks]

(b) Hence or otherwise show that the sequence $\left\{\sqrt{\frac{4n+1}{n+4}},\ n=0,\ 1,\ 2,\ldots\right\}$ is convergent.

Note that $\frac{4(n+1)+1}{(n+1)+4}-\frac{4n+1}{n+4}=\frac{15}{(n+5)(n+4)}>0$ so $\frac{4n+1}{n+4}$ is monotone increasing.

Also $4-\frac{4n+1}{n+4}=\frac{3}{n+4}>0$ so $\frac{4n+1}{n+4}$ is bounded above by 4.

Since $\sqrt{x}$ is a monotone increasing function , $x<y\Rightarrow\sqrt{x}<\sqrt{y}$ so $\sqrt{\frac{4n+1}{n+4}}$ is monotone increasing, and bounded above by $\sqrt{4}=2$. Hence it is convergent.

[4 marks]

2. Consider the set Σ_2 consisting of all sequences on the two elements 0 and 1, with the distance function between any two sequences $s=s_0,s_1,s_2,\ldots$ and $t=t_0,t_1,t_2,\ldots$ defined by $d(s,t)=\sum_{i=0}^{\infty}\frac{|s_i-t_i|}{2^{i+1}}$.

(a) Show (Σ_2,d) is a metric space i.e. that d is a metric distance function obeying **9.5.9** on p 123 of "Chapter 0".

(i) $d(s,t)\geq 0$ since $|s_i-t_i|\geq 0$, $d(s,t)=0 \Leftrightarrow \forall i\ |s_i-t_i|=0\Leftrightarrow s=t$

(ii) $d(s,t)=\sum_{i=0}^{\infty}\frac{|s_i-t_i|}{2^{i+1}}=\sum_{i=0}^{\infty}\frac{|t_i-s_i|}{2^{i+1}}=d(t,s)$

(iii) $d(s,t)+d(t,u)=\sum_{i=0}^{\infty}\frac{|s_i-t_i|}{2^{i+1}}+\sum_{i=0}^{\infty}\frac{|t_i-u_i|}{2^{i+1}}=\sum_{i=0}^{\infty}\frac{|s_i-t_i|+|t_i-u_i|}{2^{i+1}}\geq\sum_{i=0}^{\infty}\frac{|s_i-u_i|}{2^{i+1}}=d(s,u)$

Since $\forall i|s_i-t_i|+|t_i-u_i|\geq|s_i-u_i|$.

[4 marks]

(b) Show that
$$d(s,t)<\frac{1}{2^n}\Rightarrow s_i=t_i,\ i=0,\ \ldots\ ,n-1,$$
$$s_i=t_i,\ i=0,\ \ldots\ ,n-1\Rightarrow d(s,t)\leq\frac{1}{2^n}$$
.

If $\exists\ i\leq n-1:s_i\neq t_i,\ \Rightarrow d(s,t)\geq\frac{1}{2^{i+1}}\geq\frac{1}{2^n}$,

conversely $\forall\ i\leq n-1:s_i=t_i,\ \Rightarrow d(s,t)\leq 0+\ \ldots\ +\sum_{i=n}^{\infty}\frac{|s_i-t_i|}{2^{i+1}}\leq\sum_{i=n}^{\infty}\frac{1}{2^{i+1}}=\frac{1}{2^n}$

[3 marks]

(c) Show that $\left(\Sigma_2, d\right)$ is complete i.e. that every Cauchy sequence is convergent. **[4 marks]**

Let $\left\{s^i = s_0^i, s_1^i, s_2^i \ \ ..., \ i = 0,1,2 \ \ ... \right\}$ be a Cauchy sequence of sequences in Σ_2.
then $\forall p > 0, \ \exists N_p: \ \ m,n > N, \ d(s^m, s^n) < \dfrac{1}{2^p} \Rightarrow s_i^m = s_i^n, \ i = 0, \ ... \ , p-1.$
Thus for each p we can define a unique $s_{p-1}^L = s_{p-1}^m = s_{p-1}^n$.
So define $\left\{s^L = s_1^L, s_2^L, \ ...,\right\}$ inductively for each p. Then $\left\{s^i \ \ i = 0,1,2 \ \ ... \right\} \to \left\{s^L\right\}$ since
$\forall \varepsilon > 0, \ \exists p : \dfrac{1}{2^p} < \varepsilon, \ \exists N_p: \ \ n > N, \ s_i^L = s_i^n, \ i = 0, \ ... \ , p-1 \Rightarrow d(s^L, s^n) \le \dfrac{1}{2^{p+1}} < \dfrac{1}{2^p} = \varepsilon$

3. A metric space is called compact if every sequence in the space has a convergent subsequence.
A subset of **R** is compact if and only if it is closed an bounded.

(a) Use the fact that every bounded sequence has a convergent subsequence (see sequence notes for a proof of this) to show that [0,1] is compact.

Since [0,1] is bounded, so is any sequence $\left\{x_i \ \ i = 0,1,2 \ \ ... \right\}$ in [0,1]. Since any bounded sequence has a convergent subsequence, so does any sequence in [0,1]. Also since v is closed this sequence must tend to a limit in [0,1]. Otherwise $\left\{x_i \ \ i = 0,1,2 \ \ ... \right\} \to L \notin [0,1]$. Say $L > 1$, then $L = 1 + d$.
but $x_i \in [0,1]$, so $\left|x_i - L\right| \ge d$ contradicting convergence to L. **[4 marks]**

(b) Give examples of a sequences in (0,1) and **R** which have no convergent subsequence in their respective sets.

$\left\{\dfrac{1}{n}\right\} \to 0 \notin (0,1), \ \ \{n\} \to \infty \notin \mathbf{R}$. These are both monotone and thus have every subsequence tending to the same "limit", so neither has a subsequence convergent in their respective set. **[2 marks]**

(c) Show that $\left(\Sigma_2, d\right)$ is compact.

We show an arbitrary sequence $\left\{s^i = s_0^i, s_1^i, s_2^i \ \ ..., \ i = 0,1,2 \ \ ... \right\}$ in Σ_2 has a convergent subsequence.
Consider the first terms s_0^i in each of the sequence $s^i, \ i = 0,1,2$ ... in the sequence of sequences $\left\{s^i\right\}$.
Either there is an infinite number of both 0s and 1s or there is a finite number of one of them.
If there are a finite number of 0s pick the subsequence $\left\{s^{i1}\right\}$ consisting of all those sequences $s^i : s_0^i = 1$.
Otherwise pick the corresponding subsequence $\left\{s^{i0}\right\}$, consisting of all those sequences $s^i : s_0^i = 0$.
Call this chosen subsequence $\left\{s^{in_0}\right\}$ and define $n_0 = 0, \ \ or \ \ 1$ to be the first term in a sequence $\left\{n_i\right\}$.

Proceed inductively to define subsequence $\left\{s^{in_0n_1}\right\}$ of $\left\{s^{in_0}\right\}$ and $n_1 = 0, \ \ or \ \ 1$ to be the second term in $\left\{n_i\right\}$,
and in turn $\left\{s^{in_0n_1 \ \cdots \ n_p}\right\} \subseteq \left\{s^{in_0n_1 \ \cdots \ n_{p-1}}\right\}$ and the($p+1$)-th term in $\left\{n_i\right\}$.
Now consider the sequence $\left\{\bar{s}^i = s_0^{in_0}, s_1^{in_0n_1}, s_2^{in_0n_1n_2} ..., \ i = 0,1,2 \ \ ... \right\}$.

This is also clearly a subsequence of $\left\{s^i\right\}$ since all its terms occur at successively later stages of $\left\{s^i\right\}$ since the second term of $\left\{s^{in_0n_1}\right\}$ is later in $\left\{s^i\right\}$ than the first term of $\left\{s^{in_0}\right\}$, since $\left\{s^{in_0n_1}\right\}$ is itself a subsequence of $\left\{s^{in_0}\right\}$. Also since $\bar{s}_p^i = s_2^{in_0n_1 \ \cdots \ n_p} \in \left\{s^{in_0n_1 \ \cdots \ n_p}\right\}, \left\{\bar{s}^i\right\} \to \left\{n_i\right\}$. **[5 marks]**