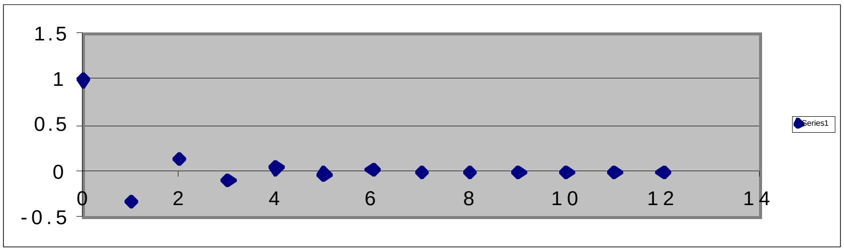
DEPARTMENT OF MATHEMATICS

MATHS 255FC	Assignment 7 solutions	Due 2nd May 2001

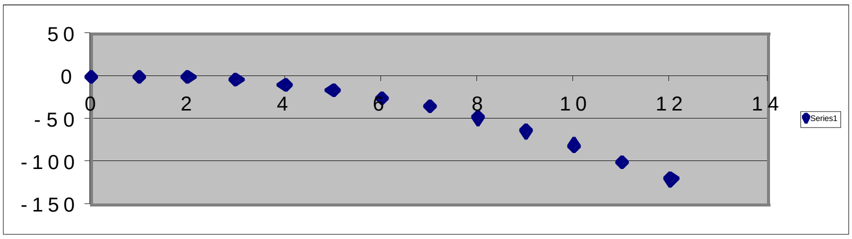
1. Determine whether the following sequences have distinct terms, are increasing, decreasing, monotonic or have an upper or lower bound. If the sequence has an upper bound find the least upper bound.

(i) $u_n = \dfrac{(-2)^n}{3^n(n+1)}$, $n = 0, 1, 2, \ldots$



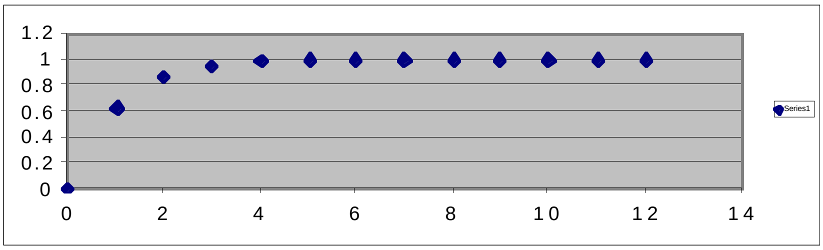
Oscillating because of the $(-2)^n$ term so neither increasing nor decreasing. Distinct elements because their absolute value is strictly decreasing. Bounded above and below. LUB=1, (NOTE: $\lim_{n\to\infty} u_n=0$). **[2]**

(ii) $u_n = n(2-n)$, $n = 0, 1, 2, \ldots$



Neither monotone nor of distinct terms because $u_n = 0$, $u_1 = 2$, $u_2 = 0$ (however strictly decreasing from then on). Unbounded below. (So: $\lim_{n\to\infty} u_n=-\infty$). LUB = 2. **[2]**

(iii) $u_n = 1 - e^{-n}$, $n = 0, 1, 2, \ldots$



Strictly monotone increasing and bounded above with LUB = 1. (NOTE: $\lim_{n\to\infty} u_n=1$). **[2]**

2. Show that the following sets of numbers form (infinite) groups with respect to ordinary multiplication:

(i) $\{2^k : k = 0, \pm1, \pm2, \ldots\}$

$2^k.2^l = 2^{(k+l)} \in G$, $2^{-k}.2^k = 2^0 = 1 \in G$ so both identity and all inverses exist. Thus a group because associativity is inherited from the reals. **[4]**

(ii) $\left\{\dfrac{1+2m}{1+2n} : m,n = 0, \pm1, \pm2, \ldots\right\}$

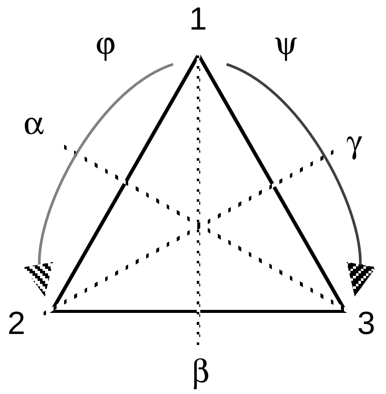
$\dfrac{1+2m}{1+2n}.\dfrac{1+2p}{1+2q} = \dfrac{1+2(m+p+mp)}{1+2(n+q+nq)} \in G$, $\dfrac{1+2m}{1+2n}.\dfrac{1+2n}{1+2m} = 1 \in G$ so both identity and all inverses exist.
Thus a group because associativity is inherited from the reals. **[4]**

3. Consider the symmetries of an equilateral triangle.

(i) Show this group is isomorphic to the group of permutations of three elements S_3.

The Cayley table for the group illustrated on page 5 of the Groups notes is shown below reordered

	e	φ	ψ	α	β	γ
e	e	φ	ψ	α	β	γ
φ	φ	ψ	e	γ	α	β
ψ	ψ	e	φ	β	γ	α
α	α	β	γ	e	φ	ψ
β	β	γ	α	ψ	e	φ
γ	γ	α	β	φ	ψ	e



The isomorphism between S_3 and the group D_3 of symmetries of the triangle is defined by each of α, β, γ with the reflections in the three axes shown in the diagram. φ, ψ are associated with rotations of $\frac{2\pi}{3}$ and $\frac{4\pi}{3} = \frac{-2\pi}{3}$ respectively as shown in the triangle diagram. This is clearly an isomorphism because the symmetries permute the vertices exactly as in S_3. **[6]**

(ii) Show the subgroup consisting of rotations of the triangle is isomorphic with the additive group $\mathbf{Z}_3$ of integers modulo 3.

The first three elements in the above Cayley table i.e. the identity and the two rotations are clearly additive angles of $0, \frac{\pi}{3}, \frac{2\pi}{3}$ modulo 2π precisely the same as the additive group of 0, 1, 2 mod 3. viz

0	1	2
1	2	0
2	0	1

[2]

4. Show that $\mathbf{Z}_6$ with the zero deleted is not a multiplicative group,

x	1	2	3	4	5
1	1	2	3	4	5
2	2	4	0	2	4
3	3	0	3	0	3
4	4	2	0	4	2
5	5	4	3	2	1

This has rows of non distinct elements so cannot be a group of distinct elements by the condensation law. **[3]**

but that $\mathbf{Z}_7$ with the zero deleted is a multiplicative group isomorphic to the additive group $\mathbf{Z}_6$.

$(Z_6, +) \sim$

0	1	2	3	4	5
1	2	3	4	5	0
2	3	4	5	0	1
3	4	5	0	1	2
4	5	0	1	2	3
5	0	1	2	3	4

$(Z_7 \sim 0, x) \sim$

1	2	3	4	5	6
2	4	6	1	3	5
3	6	2	5	1	4
4	1	5	2	6	3
5	3	1	6	4	2
6	5	4	3	2	1

or

1	3	2	6	4	5
3	2	6	4	5	1
2	6	4	5	1	3
6	4	5	1	3	2
4	5	1	3	2	6
5	1	3	2	6	4

Now the isomorphism clearly is: $f(0) = 1, f(3) = 6, f(2) = 2, f(4) = 4, f(5) = 5, f(1) = 3$ **[5]**