1. (a) We prove this by induction. Note that, because we want to know that the result holds for all $n \geq 0$, we have $n = 0$ for the base instead of $n = 1$.

Base: When $n = 0$ we have $10^0 = 1 \equiv 1 \pmod 3$, so the result holds for $n = 0$.

Inductive step: Suppose $n \geq 0$ and $10^n \equiv 1 \pmod 3$. Then $3 \mid 10^n - 1$, so there is some $x \in \mathbb{Z}$ with $10^n - 1 = 3x$. But then

$$\begin{aligned} 10^{n+1} - 1 &= 10^{n+1} - 10^n + 10^n - 1 = 10^n(10-1) + 3x \\ &= 3(3 \cdot 10^n + x), \end{aligned}$$

so $3 \mid 10^{n+1} - 1$, so $10^{n+1} \equiv 1 \pmod 3$, as required.

Hence, by induction, $10^n \equiv 1 \pmod 3$ for all $n \geq 0$.

(b) We have $n = \sum_{i=0}^{k} a_i 10^i$. So we have $n \equiv \sum_{i=0}^{k} a_i 10^i \pmod 3$. But we also have $10^i \equiv 1 \pmod 3$ for each i, so $n \equiv \sum_{i=0}^{k} a_i \cdot 1 \pmod 3$, i.e.

$$n \equiv \sum_{i=0}^{k} a_i \pmod 3,$$

as required.

(c) Suppose $3 \mid n$. So there is some $x \in \mathbb{Z}$ such that $n = 3x$. By part (b), we have $n \equiv \sum_{i=0}^{k} a_i \pmod 3$, so there is some $y \in \mathbb{Z}$ such that $n - \sum_{i=0}^{k} a_i = 3y$. But then $3x - \sum_{i=0}^{k} a_i = 3y$, so $\sum_{i=0}^{k} a_i = 3(x - y)$, so $3 \mid \sum_{i=0}^{k} a_i$.

Conversely, suppose $3 \mid \sum_{i=0}^{k} a_i$. Then $\sum_{i=0}^{k} a_i = 3z$ for some $z \in \mathbb{Z}$. As above, $n - \sum_{i=0}^{k} a_i = 3y$ for some $y \in \mathbb{Z}$, so $n = 3y + 3z) = 3(y + z)$, so $3 \mid n$.

2. First we find y so that $15y \equiv 1 \pmod{37}$, by using the Euclidean Algorithm for 37 and 15:

37	1	0
15	0	1
7	1	−2
1	−2	5
0	15	−37

From this we see that $37 \cdot (-2) + 15 \cdot 5 = 1$, so $15 \cdot 5 \equiv 1 \pmod{37}$, in other words $\overline{15} \cdot_{37} \overline{5} = \overline{1}$.

Now we multiply both sides of the equation $\overline{16} = \overline{15} \cdot_{37} \overline{x}$ by $\overline{5}$ to get

$$\begin{aligned} \overline{5} \cdot_{37} \overline{16} &= \overline{5} \cdot_{37} \overline{15} \cdot_{37} \overline{x} \\ &= \overline{1} \cdot_{37} \overline{x}, \end{aligned}$$

so the solution is $\overline{x} = \overline{5} \cdot_{37} \overline{16} = \overline{5 \cdot 16} = \overline{80} = \overline{6}$.

3. We use Euclid's Algorithm. First we divide $x^3 + x^2 - 4x - 4$ into $x^4 + 3x^3 + 3x^2 + 3x + 2$:

$$\begin{array}{rrrrrrr}
 & & x & +2 & & & \\
\cline{2-7}
x^3 + x^2 - 4x - 4 & \big) & x^4 & +3x^3 & +3x^2 & +3x & +2 \\
 & & x^4 & +x^3 & -4x^2 & -4x & \\
\cline{2-6}
 & & & 2x^3 & +7x^2 & +7x & +2 \\
 & & & 2x^3 & +2x^2 & -8x & -8 \\
\cline{4-7}
 & & & & 5x^2 & +15x & +10
\end{array}$$

So $x^4+3x^3+3x^2+3x+2 = (x+2)(x^3+x^2-4x-4)+(5x^2+15x+10)$. Next we divide $5x^2+15x+10$ into $x^3 + x^2 - 4x - 4$:

$$\begin{array}{rrrrrr}
 & & \frac{1}{5}x & -\frac{2}{5} & & \\
\cline{2-6}
5x^2 + 15x + 10 & \big) & x^3 & +x^2 & -4x & -4 \\
 & & x^3 & +3x^2 & +2x & \\
\cline{2-5}
 & & & -2x^2 & -6x & -4 \\
 & & & -2x^2 & -6x & -4 \\
\cline{4-6}
 & & & & & 0
\end{array}$$

So we have $x^3+x^2-4x-4 = (\frac{1}{5}x - \frac{2}{5})(5x^2+15x+10)$. The last non-zero remainder is $5x^2+15x+10$, so this is a greatest common divisor.

[Note that any non-zero constant multiple of this is also a greatest common divisor. One convention we use is to choose the *monic* polynomial, in other words the polynomial which has 1 as the coeffieicent for the term of highest degree. So we would say that *the* greatest common divisor of $x^4 + 3x^3 + 3x^2 + 3x + 2$ and $x^3 + x^2 - 4x - 4$ is $x^2 + 3x + 2$.]

4. (a) Since the degree of $ax + b$ is 1, if it is reducible then we can write it as $ax + b = p(x)q(x)$, where $p(x)$ and $q(x)$ both have degree less than 1. But then both $p(x)$ and $q(x)$ are constant, so $p(x)q(x)$ is constant, so it cannot equal $ax + b$.

(b) Since the degree of x^2+1 is 2, if it is reducible then we can write it as $x^2+1 = p(x)q(x)$, where $p(x)$ and $q(x)$ both have degree less than 2. Since $\deg p(x) + \deg q(x) = 2$, $p(x)$ and $q(x)$ must both have degree 1. So $p(x) = ax + b$ for some $a, b \in \mathbb{R}$ with $a \neq 0$. Then

$$x^2 + 1 = a\left(x - \tfrac{-b}{a}\right) q(x),$$

so $\left(x - \frac{-b}{a}\right) \mid x^2 + 1$. By the Factor Theorem, this implies that $\left(\frac{-b}{a}\right)^2 + 1 = 0$, which is impossible as $\left(\frac{-b}{a}\right)^2 \geq 0$.