1. (a) Suppose that $g \circ f$ is onto and g is one-to-one. Let $b \in B$. Then $c = g(b) \in C$, and $g \circ f : A \to C$ is onto, so there is some $x \in A$ with $(g \circ f)(x) = c$. Then $g(f(x)) = c = g(b)$, and g is one-to-one, so $f(x) = b$. Hence f is onto.

(b) Let $f : [0, \infty) \to \mathbb{R}$ and $g : \mathbb{R} \to [0, \infty)$ be given by $f(x) = \sqrt{x}$ and $g(x) = x^2$.[1] Then, for any $x \in [0, \infty)$, $(g \circ f)(x) = (\sqrt{x})^2 = x$. Hence $g \circ f$ is onto. However, f is not onto as there is no $x \in [0, \infty)$ with $\sqrt{x} = -1$.

2. Suppose f is order preserving. Let $x, y \in \mathbb{R}$ with $f(x) = f(y)$. Then $f(x) \leq f(y)$ and $f(y) \leq f(x)$, so $x \leq y$ and $y \leq x$, so $x = y$. Hence f is one-to-one.

The converse does not hold: for example, the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = -x$ is one-to-one but is not order-preserving.

3. (a) Suppose f and g are both foo functions. Let $x \in R$. Then

$$\begin{aligned}(f \circ g)(x+1) &= f(g(x+1)) \\ &= f(g(x)+1) && \text{since } g \text{ is a foo function} \\ &= f(g(x))+1 && \text{since } f \text{ is a foo function} \\ &= (f \circ g)(x)+1,\end{aligned}$$

so $f \circ g$ is a foo function.

(b) Suppose first that f is a foo function. Let $x \in \mathbb{R}$. Then $x - 1 \in \mathbb{R}$, so since f is a foo function we have $f((x-1)+1) = f(x-1)+1$, in other words $f(x) = f(x-1)+1$, so $f(x-1) = f(x)-1$.

Conversely, suppose that for every $x \in \mathbb{R}$ we have $f(x-1) = f(x) - 1$. Let $x \in \mathbb{R}$. Put $t = x + 1$. Then, by hypothesis, $f(t-1) = f(t) - 1$, so $f((x+1)-1) = f(x+1) - 1$, in other words $f(x) = f(x+1) - 1$, so $f(x+1) = f(x) + 1$. Hence f is a foo function.

[Note: if may be tempting to say that, because $f(x+1) = f(x) + 1$, we have $f(x + (-1)) = f(x) + (-1)$. In other words, we have $f(x+1) = f(x) + 1$, so "obviously" for any a we have $f(x+a) = f(x) + a$. This is not OK, however: while x is a variable which can be replaced with any value in $\mathbb{R}$, 1 is a fixed number, which we can't just vary freely.]

(c) Suppose f is a foo function and g is a bar function. Then

$$\begin{aligned}(f \circ g)(x+1) &= f(g(x+1)) \\ &= f(g(x)-1) && \text{since } g \text{ is a bar function} \\ &= f(g(x))-1 && \text{by part (b), since } f \text{ is a foo function} \\ &= (f \circ g)(x)-1.\end{aligned}$$

Hence $f \circ g$ is a bar function.

[1] Note that $\sqrt{x}$ always denotes the *positive* square root of x.

4. For $n \in \mathbb{N}$ let P_n be the statement that $n^3 + 5n$ is a multiple of 6.

Base: P_1 is the statement that $1 + 5$ is a multiple of 6, which is true.

Inductive step: Suppose $n \in \mathbb{N}$ and P_n is true. So there is some $k \in \mathbb{Z}$ such that $n^3 + n = 6k$. Then

$$\begin{aligned}(n+1)^3 + 5(n+1) &= n^3 + 3n^2 + 3n + 1 + 5n + 5 \\ &= (n^3 + 5n) + 3n^2 + 3n + 6 \\ &= (n^3 + 5n) + 3(n^2 + n) + 6.\end{aligned}$$

Now, $n^2 + n$ is even, so $n^2 + n = 2l$ for some $l \in \mathbb{Z}$, so we have

$$\begin{aligned}(n+1)^3 + 5(n+1) &= (n^3 + 5n) + 3(n^2 + n) + 6 \\ &= 6k + 3 \cdot 2l + 6 \\ &= 6(k + l + 1),\end{aligned}$$

so $(n+1)^3 + 5(n+1)$ is also a multiple of 6, in other words P_{n+1} is also true.

Hence, by induction, P_n is true for all $n \in \mathbb{N}$.