1. We must show that $\preccurlyeq$ is reflexive, antisymmetric and transitive.

Reflexive: Let $(a, b) \in A^2$. Then $a \leq a$ and $b \leq b$, so $(a, b) \preccurlyeq (a, b)$.

Antisymmetric: Let $(a, b), (c, d) \in A^2$ with $(a, b) \preccurlyeq (c, d)$ and $(c, d) \preccurlyeq (a, b)$. Then $a \leq c$ and $b \leq d$, and $c \leq a$ and $d \leq b$. So $a \leq c \leq a$, so $a = c$, and $b \leq d \leq b$, so $b = d$. Hence $(a, b) = (c, d)$.

Transitive: Let $(a, b), (c, d), (e, f) \in A^2$ with $(a, b) \preccurlyeq (c, d)$ and $(c, d) \preccurlyeq (e, f)$. Then $a \leq c$ and $b \leq d$, and $c \leq e$ and $d \leq f$. So $a \leq c \leq e$, so $a \leq e$, and $b \leq d \leq f$, so $b \leq f$. Hence $(a, b) \preccurlyeq (e, f)$.

2. [This is an "iff" proof, so we must prove two implications.]

Suppose first that l is a least upper bound for A. Let $\varepsilon \in \mathbb{R}$ with $\varepsilon > 0$. Put $b = l - \varepsilon$. Then $b < l$, and l is a least upper bound for A, so b is not an upper bound for A. So there is some $a \in A$ with $a \not\leq b$, and then $b < a$. We know that $a \leq l$, since l is an upper bound for A, so we have $a \in (b, l]$, i.e. $A \in (l - \varepsilon, l]$. Thus $(l - \varepsilon, l] \cap A \neq \varnothing$.

Conversely, suppose that for every $\varepsilon \in \mathbb{R}$ with $\varepsilon > 0$ we have $(l - \varepsilon, l] \cap A \neq \varnothing$. We must show that l is a least upper bound for A. We know that l is an upper bound for A, so we must show that if b is an upper bound for A then $l \leq b$. Suppose, for a contradiction, that there is an upper bound b for A with $b < l$. Put $\varepsilon = l - b$. Then $\varepsilon \in \mathbb{R}$ and $\varepsilon > 0$, so by hypothesis we have $(l - \varepsilon, l] \cap A \neq \varnothing$. Choose some $a \in (l - \varepsilon, l] \cap A$. Then we have $a \in A$ and $b = l - \varepsilon < a$, contradicting the assumption that b is an upper bound for A. So there is no such b, in other words L is a least upper bound for A.

3. We must show that $\sim$ is reflexive, symmetric and transitive.

Reflexive: Let $(a, b) \in A$. Then $ab = ba$, so $(a, b) \sim (a, b)$.

Symmetric: Let $(a, b), (c, d) \in A$ with $(a, b) \sim (c, d)$. Then $ad = bc$, so $cb = da$, so $(c, d) \sim (a, b)$.

Transitive: Let $(a, b), (c, d), (e, f) \in A$ with $(a, b) \sim (c, d)$ and $(c, d) \sim (e, f)$. Then

$$ad = bc \tag{1}$$

$$cf = de \tag{2}$$

Multiplying (1) by f and (2) by b we get $adf = bcf$ and $bcf = bde$, so $adf = bde$. Since we also have $d \in \mathbb{N}$ we have $d \neq 0$, so we can divide by d to get $af = be$, so $(a, b) \sim (e, f)$.

4. Suppose first that F is one-to-one. We want to show that f is onto, so let $b \in B$. Then $\{b\} \neq \varnothing$, so since F is one-ot-one we have $F(\{b\}) \neq \varnothing$. Now, $F(\varnothing) = \{x \in A : f(x) \in \varnothing\}$, so $F(\varnothing) = \varnothing$. Hence we have $F(\{b\}) \neq \varnothing$, so there is some $a \in A$ with $a \in F(\{b\})$, in other words with $f(a) \in \{b\}$. But then $f(a) = b$, as required.

Conversely, suppose that f is onto. Let $X, Y \in \mathcal{P}(B)$ with $F(X) = F(Y)$. We wish to show that $X = Y$. So let $x \in X$. Then $x \in B$, and f is onto, so there is some $a \in A$ with $f(a) = x$. Then $f(a) \in X$, so $a \in F(X) = F(Y)$, so $a \in F(Y)$, so $f(a) \in Y$, so $x \in Y$. Hence $X \subseteq Y$. Conversely, let $y \in Y$. Again, $y = f(c)$ for some $c \in A$, so we have $f(c) \in Y$, so $c \in F(Y) = F(X)$, so $f(c) \in X$, so $y \in X$, and hence $Y \subseteq X$. Combining these, we have $X = Y$. Hence F is one-to-one.