1. (a) Let $X = \{\, a + b\sqrt{2} : a, b \in \mathbb{Z} \,\}$. We will show that X is a smallest subring of $\mathbb{R}$ containing $\sqrt{2}$.

- X is a subring of $\mathbb{R}$: we certainly have $X \subseteq \mathbb{R}$, so there are two conditions to check:
  (1) let $a \in \mathbb{Z}$. Then $a = a + 0\sqrt{2} \in X$. So $\mathbb{Z} \subseteq X$.
  (2) let $x, y \in X$. Then $x = a + b\sqrt{2}$ and $y = c + d\sqrt{2}$ for some $a, b, c, d \in \mathbb{Z}$. But then

$$x + y = (a + b\sqrt{2}) + (c + d\sqrt{2}) = (a + c) + (b + d)\sqrt{2} \in X,$$

  since $a + c, b + d \in \mathbb{Z}$. Similarly we have

$$\begin{aligned} xy &= (a + b\sqrt{2})(c + d\sqrt{2}) \\ &= ac + ad\sqrt{2} + bc\sqrt{2} + bd(\sqrt{2})^2 \\ &= (ac + 2bd) + (ad + bc)\sqrt{2} \in X, \text{ and} \\ -x &= (-a) + (-b)\sqrt{2} \in X. \end{aligned}$$

- $\sqrt{2} \in X$: we have $\sqrt{2} = 0 + 1\sqrt{2} \in X$.
- if S is a subring of $\mathbb{R}$ with $\sqrt{2} \in S$ then $X \subseteq S$: let S be a subring of $\mathbb{R}$ with $\sqrt{2} \in S$. Let $x \in X$. Then $x = a + b\sqrt{2}$ for some $a, b \in \mathbb{Z}$. Since $\mathbb{Z} \subseteq S$ we have $a, b \in S$. Since we also have $\sqrt{2} \in S$ we have $b\sqrt{2} \in S$, so we have $a + b\sqrt{2} \in S$, in other words $x \in S$. Thus $X \subseteq S$.

(b) Let X and Y be smallest subrings of $\mathbb{R}$ contining $\sqrt{2}$. Then $X \subseteq S$ for any subring of $\mathbb{R}$ containing $\sqrt{2}$: in particular, $X \subseteq Y$. Similarly, $Y \subseteq S$ for any subring of $\mathbb{R}$ containing $\sqrt{2}$: in particular, $Y \subseteq X$. Hence $X = Y$.

2. (1) $\Longrightarrow$ (2)**:** Suppose $A \subseteq B$. Let $x \in A \cup B$. Then $x \in A$ or $x \in B$.

Case 1: $x \in A$. Then since $A \subseteq B$ we have $x \in B$.

Case 2: $x \in B$. Then we have $x \in B$.

So in either case we have $x \in B$. Thus $A \cup B \subseteq B$.

Conversely, let $y \in B$. Then $y \in A$ or $y \in B$, so $y \in A \cup B$. Thus $B \subseteq A \cup B$.

Combining these, we have $A \cup B = B$.

(2) $\Longrightarrow$ (3)**:** Suppose $A \cup B = B$. Suppose, for a contradiction, that $A \setminus B \neq \varnothing$. Let $x \in A \setminus B$. Then $x \in A$ and $x \notin B$. Since $x \in A$, we have $x \in A \cup B = B$, so $x \in B$. But this contradicts the earlier assertion that $x \notin B$. So there is no such x, i.e. $A \setminus B = \varnothing$.

(3) $\Longrightarrow$ (1)**:** We'll prove the contrapositive. Suppose that $A \nsubseteq B$. Then it is not true that every element of A is an element of B, in other words, there exists some $x \in A$ such that $x \notin B$. But then $x \in A \setminus B$, so $A \setminus B \neq \varnothing$. Hence, by contraposition, if $A \setminus B = \varnothing$ then $A \subseteq B$.

3. (a) Let $x \in C \setminus (A \cup B)$. Then $x \in C$ and $x \notin A \cup B$. Since $x \notin A \cup B$ we have $x \notin A$ and $x \notin B$. So we have $x \in C$ and $x \notin A$, so $x \in C \setminus A$. We also have $x \in C$ and $x \notin B$, so $x \in C \setminus B$. Putting these together we have $x \in (C \setminus A) \cap (C \setminus B)$. Hence $C \setminus (A \cup B) \subseteq (C \setminus A) \cap (C \setminus B)$.

Conversely, let $y \in C \setminus A) \cap (C \setminus B)$. Then $y \in C \setminus A$ and $y \in C \setminus B$, so $y \in C$ and $y \notin A$, and $y \in C$ and $y \notin B$. Since $y \notin A$ and $y \notin B$, $y \notin A \cup B$, so we have $y \in C$ and $y \notin A \cup B$, so $y \in C \setminus (A \cup B)$. Hence $(C \setminus A) \cap (C \setminus B) \subseteq C \setminus (A \cup B$.

Combining these, we have $C \setminus (A \cup B) = (C \setminus A) \cap (C \setminus B)$.

(b) Let $x \in B \setminus (B \setminus A)$. Then $x \in B$ and $x \notin B \setminus A$. Since $x \notin B \setminus A$, it is not true that $x \in B$ and $x \notin A$, so we have $x \notin B$ or $x \in A$. We already know that $x \in B$, so we must have $x \in A$. Thus we have $x \in A \cap B$. Hence $B \setminus (B \setminus A) \subseteq A \cap B$.

Conversely, let $y \in A \cap B$. Then $y \in A$ and $y \in B$. Since $y \in A$, $y \notin B \setminus A$, so $y \in B \setminus (B \setminus A)$. Hence $A \cap B \subseteq B \setminus (B \setminus A)$.

Combining these, we have $B \setminus (B \setminus A) = A \cap B$.

4. (a) Let $X \in \mathcal{P}(A \cap B)$. Then $X \subseteq A \cap B$. Let $x \in X$. Then $x \in A \cap B$, so $x \in A$ and $x \in B$. Thus every element of X is an element of A, so $X \subseteq A$, and also every element of X is an element of B, so $X \subseteq B$. Thus $X \in \mathcal{P}(A)$ and $X \in \mathcal{P}(B)$, so $X \in \mathcal{P}(A) \cap \mathcal{P}(B)$. Hence $\mathcal{P}(A \cap B) \subseteq \mathcal{P}(A) \cap \mathcal{P}(B)$.

Conversely, let $Y \in \mathcal{P}(A) \cap \mathcal{P}(B)$. Then $Y \in \mathcal{P}(A)$ and $Y \in \mathcal{P}(B)$, so $Y \subseteq A$ and $Y \subseteq B$. Let $y \in Y$. Then $y \in A$ (since $Y \subseteq A$), and $y \in B$ (since $Y \subseteq B$), so $y \in A \cap B$. Thus $Y \subseteq A \cap B$. Hence $\mathcal{P}(A) \cap \mathcal{P}(B) \subseteq \mathcal{P}(A \cap B)$.

Combining these, we get $\mathcal{P}(A \cap B) = \mathcal{P}(A) \cap \mathcal{P}(B)$.

(b) Let $A = \{1, 2\}$ and $B = \{3\}$. Then $\{1, 3\} \in \mathcal{P}(A \cup B)$, but $\{1, 3\} \notin \mathcal{P}(A) \cup \mathcal{P}(B)$, so in this example we have $\mathcal{P}(A \cup B) \neq \mathcal{P}(A) \cup \mathcal{P}(B)$.