Differentiation

Recall that $f'(a) = \lim_{h\to 0} \dfrac{f(a+h)-f(a)}{h}$ is the *derivative of f at a* (provided the limit exists. To avoid problems we take x to be in an *open* interval on which f is defined. The derivative function f' is the function whose value at a is $f'(a)$.

There are classical examples of continuous but nowhere differentiable functions. However in the other direction we have the following result:

Theorem f differentiable at $a \implies f$ continuous at a.

Rules for derivatives of sums and products are just particular cases of previous results about limits. Higher derivatives are defined recursively, i.e. $f^{(n)} = (f^{(n-1)})'$. Most common functions are infinitely differentiable (of class C^∞) except possibly at a finite or countable set of points.

Chain Rule If f is differentiable at x and g is differentiable at $f(x)$ then the composite function $g \circ f$ is differentiable at x and

$$(g \circ f)'(x) = g'[f(x)]f'(x).$$

Proof Note first that f is differentiable at x if and only if there is a number L and a function $E(x,h)$ such that

$$f(x+h) = f(x) + Lh + hE(x,h), \quad \text{where } E(x,h) \to 0 \text{ as } x \to 0.$$

In fact if f is differentiable at x we can take $E(x,h) = \dfrac{f(x+h)-f(x)}{h} - f'(x)$ and we can make $E(x,h)$ continuous at 0 by defining $E(x,0) = 0$.

Now let $k = k(h) = f(x+h) - f(x)$ and let $y = f(x)$. Then

$$g[f(x+h)] - g[f(x)] = g(y+k) - g(y) = g'(y)k + kE(y,k),$$

where $E(y,k) \to 0$ as $k \to 0$ (and $E(y,0) = 0$ by definition). So

$$\frac{g[(f(x+h)] - g[(f(x)]}{h} = g'[f(x)]\frac{f(x+h)-f(x)}{h} + \frac{f(x+h)-f(x)}{h}E[f(x),k(h)].$$

Now take limits as $h \to 0$ and use the fact that the functions E and k are continuous at 0 and take the value 0 there.

The following result is crucial in the application of differential calculus to maximum and minimum problems:

Theorem Let f be defined on an open interval I, and assume that f has a relative maximum or minimum at an *interior* point p of I. If the derivative $f'(p)$ exists then $f'(p) = 0$.

Proof Define a function Q on I by $Q(x) = \dfrac{f(x) - f(p)}{x - p}$ for $x \neq p$ and $Q(p) = f'(p)$. Since $f'(p)$ exists, $Q(x) \to Q(p)$ as $x \to p$ and so Q is continuous at p. We must show that $Q(p) = 0$.

Suppose $Q(p) > 0$. By the sign-preserving property of continuous functions there is an interval about p in which $Q(x)$ is positive. So the numerator and denominator of $Q(x)$ have the same sign for all $x \neq 0$ in this interval, i.e.

$$f(x) > f(p) \text{ for } x > p \quad \text{and} \quad f(x) < f(p) \text{ for } x < p.$$

But this contradicts the asumption that f has an extremum at p. Hence $Q(p) > 0$ is impossible. Similarly $Q(p) < 0$ is impossible. So $Q(p) = 0$.

Rolles Theorem If f is continuous on $[a, b]$, differentiable on (a, b) and $f(a) = f(b)$, then there is at least one point p in (a, b) such that $f'(p) = 0$.

Proof Since f is continuous on $[a, b]$ it follows from the Extreme Value Theorem that it has a maximum and a minimum in $[a, b]$. If there is a maximum M or minimum m at $p \in (a, b)$ then $f'(p) = 0$ by the previous theorem. If both extreme values occur at endpoints then $M = m$ since $f(a) = f(b)$ and so f is constant on $[a, b]$, i.e. $f'(p) = 0$ for any $p \in (a, b)$.

Mean Value Theorem If f is continuous on $[a, b]$ and differentiable on (a, b) then there is at least one point $p \in (a, b)$ such that $f(b) - f(a) = (b - a)f'(p)$.

Proof We apply Rolles Theorem to the function

$$g(x) = \left[\frac{f(b) - f(a)}{b - a}\right](x - a) + f(a) - f(x).$$

Remark
From the Mean Value Theorem we can deduce the standard results about the geometric interpretation of derivatives (e.g. f is strictly increasing on $[a, b]$ if $f'(x) > 0$ for all $x \in (a, b)$) and thus the first and second derivative tests for extrema.

Another consequence is that f is constant on $[a, b]$ if $f'(x) = 0$ on (a, b). It follows that if g is another function with the same properties as f and $f'(x) = g'(x)$ for all $x \in (a, b)$ then f and g differ only by a constant. This is a crucial point in showing that we can calculate integrals in terms of antiderivatives.