1. You may be familiar with the following test for divisibility by 3: a number is divisible by 3 iff the sum of its digits (in its decimal expansion) is divisible by 3. So for example, to test whether or not 132568 is divisible by 3, we consider the sum of its digits, which is

$$1 + 3 + 2 + 5 + 6 + 8 = 25$$

which is not divisible by 3, so $3 \nmid 132568$. Sure enough, we can verify that $132568 = 44189 \cdot 3 + 1$. We will now see why this test works.

(a) Prove that, for every $n \geq 0$, $10^n \equiv 1 \pmod 3$.

(b) Let $n \in \mathbb{N}$. Suppose that, when we write n in decimal, we have $n = a_k a_{k-1} \ldots a_2 a_1 a_0$, so that $n = \sum_{i=0}^{k} a_i 10^i$. [So for example if $n = 132568$ we have $a_5 = 1$, $a_4 = 3$, $a_3 = 2$, $a_2 = 5$, $a_1 = 6$ and $a_0 = 8$.] Show that

$$n \equiv \sum_{i=0}^{k} a_i \pmod 3.$$

(c) As a special case of (b), show that n is divisible by 3 if and only if the sum of the digits of its decimal expansion is divisible by 3.

2. Solve the equation $\overline{16} = \overline{15} \cdot_{37} \overline{x}$ in $\mathbb{Z}_{37}$.

3. Find a greatest common divisor of the polynomials $x^4 + 3x^3 + 3x^2 + 3x + 2$ and $x^3 + x^2 - 4x - 4$ in $\mathbb{R}[x]$.

4. (a) Show that any linear polynomial $ax + b$ (where $a \neq 0$) is irreducible.

(b) Show that the polynomial $x^2 + 1$ is irreducible in $\mathbb{R}[x]$.