1. Show that, for every $n \in \mathbb{N}$, $7 \mid 8^n - 1$.

2. Let a and b be natural numbers. Put

$$S = \{\, n \in \mathbb{N} : \text{for some } x, y \in \mathbb{Z},\ n = ax + by \,\}$$

Notice that $S \neq \varnothing$ (since $a = a \cdot 1 + b \cdot 0$ so $a \in S$), and therefore S has a least element, d say. We will show that $d = \gcd(a, b)$.

(a) Show that $d \mid a$. [Hint: write $a = qd + r$ with $q, r \in \mathbb{Z}$ and $0 \leq r < d$. Show that if $0 < r$ then $r \in S$, and say why this is impossible.]

(b) Show that $d \mid b$. [Use a similar method to (a).]

(c) Show that if c is a common divisor of a and b then $c \mid d$.

3. Let $a, b \in \mathbb{N}$. Suppose $a = qb + r$, with $q, r \in \mathbb{Z}$, $0 \leq r < b$.

(a) Suppose $r > 0$. Show that the common divisors of a and b are the same as the common divisors of b and r.

(b) Suppose instead that $r = 0$. Show that $\gcd(a, b) = b$.

4. (a) Find the greatest common divisor of 55 and 15.

(b) Find integers x and y such that $\gcd(55, 15) = 55x + 15y$.

(c) Find all integer solutions to the equation $55a + 15b = 20$.

(d) Find all integer solutions to the equation $55a + 15b = 23$.