1. A *subring of* $\mathbb{R}$ is a subset S of $\mathbb{R}$ such that

(1) $\mathbb{Z} \subseteq S$; and

(2) if $x, y \in S$ then $x + y \in S$, $xy \in S$ and $-x \in S$.

We define $\mathbb{Z}[\sqrt{2}]$ to be the smallest subring of $\mathbb{R}$ containing $\sqrt{2}$, in other words by declaring that

- $\mathbb{Z}[\sqrt{2}]$ is a subring of $\mathbb{R}$;
- $\sqrt{2} \in \mathbb{Z}[\sqrt{2}]$; and
- if S is a subring of $\mathbb{R}$ with $\sqrt{2} \in S$ then $\mathbb{Z}[\sqrt{2}] \subseteq S$.

To justify this definition, we have to give an existence proof and a uniqueness proof.

(a) Show that there exists a smallest subring of $\mathbb{R}$ containing $\sqrt{2}$. [To do this, we must provide a candidate, and show that it has the properties we want. The candidate is

$$X = \{\, a + b\sqrt{2} : a, b \in \mathbb{Z} \,\}.$$

Show that this set has all the properties we want.]

(b) Show that the smallest subring of $\mathbb{R}$ containing $\sqrt{2}$ is unique. [To do this, we must show that if X and Y both satisfy the conditions then $X = Y$. This is not a trick question: once you understand what is required, you can write down your answer in a few lines.]

2. Let A and B be sets. Show that the following are equivalent:

(1) $A \subseteq B$.

(2) $A \cup B = B$.

(3) $A \setminus B = \varnothing$.

3. Let A, B and C be sets. Show that the following hold.

(a) $C \setminus (A \cup B) = (C \setminus A) \cap (C \setminus B)$.

(b) $B \setminus (B \setminus A) = A \cap B$.

4. (a) Show that for any sets A and B, $\mathcal{P}(A \cap B) = \mathcal{P}(A) \cap \mathcal{P}(B)$.

(b) Give an example of sets A and B such that $\mathcal{P}(A \cup B) \neq \mathcal{P}(A) \cup \mathcal{P}(B)$.