DEPARTMENT OF MATHEMATICS

MATHS 255FC Assignment 10 Due 23rd May 2001

1. Suppose that $0 < a < 1$ and that $\{x_n\}$ is a sequence satisfying $|x_{n+1} - x_n| \le a^n$, prove that $\{x_n\}$ is Cauchy and hence convergent.

2. Let I be a real interval and $f : I \to I$ satisfy $|f(x) - f(y)| \le a|x-y|,\ x, y \in I,\ 0 < a < 1$.
Such a function is called a contraction mapping .
(i) Prove that f is continuous on I.
(ii) Let $x_1 \in I$ and define $x_{n+1} = f(x_n),\ n = 1,2,\ ....$ Use 1 above to prove that $\{x_n\}$ converges and that its limit l is a fixed point satisfying $l = f(l)$.

3. Suppose that f is continuous at every point and that $f(x) \to 0$ as $x \to \pm\infty$. Prove that f attains a maximum or a minimum value on $\mathbf{R}$.

4. A function f is continuous on the interval I and for each rational number, r in I $f(r) = r^2$. Prove $f(x) = x^2\ \ \forall x \in I$.

5. Let $f(x) = \begin{cases} x^3 - x^2, & x \in \mathbf{Q} \\ 0, & x \notin \mathbf{Q} \end{cases}$. Find any x where f is (i) continuous, (ii) differentiable.

Contraction Mappings and Fractals.

You can define a metric space (X,h) as follows: X is the set of all compact subsets of $\mathbf{R}^2$.
$d(x,A) = \text{glb}(d(x,a),\ a \in A),\ d(B,A) = \text{lub}(d(x,A),\ x \in B),\ h(B,A) = \max(d(B,A), d(A,B))$.
This distance, the Hausdorff metric is the greatest distance either set protrudes from the other.

Now we can define a contraction mapping on this space by defining a set of piecewise contraction mappings taking the unit square to a union of contracted images. The function is multiple valued on points but maps a compact set to a unique compact set. The limit of the sequence of recursive iterations is a fixed set which the function preserves under iteration. This is called the fractal attractor of the iterated function system. For example the contraction mapping consisting of three affine contractions

$$f_1\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix},\quad f_2\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1/2 \\ 0 \end{pmatrix},\quad f_3\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ 1/2 \end{pmatrix}$$

define the Sierpinski Gasket shown below.