THE UNIVERSITY OF AUCKLAND **445.255 SC**

EXAMINATION FOR BA BSc ETC 2000

MATHEMATICS

Principles of Mathematics

(Time allowed: THREE hours)

NOTE: Answer ALL the questions. All questions carry equal marks.

1. Let $f : \mathbb{N} \to \mathbb{N}$ be given by $f(x) = x^2 + 2x$.

(a) Use a **direct proof** to show that if n is even then $f(n)$ is even.

(b) Use a **proof by contraposition** to show that if $f(n)$ is even then n is even.

(c) Use a **proof by contradiction** to show that if $f(n + k)$ is odd then n is odd or k is odd.

(d) Prove that f is one-to-one. [Hint: show first that if $n < k$ then $f(n) < f(k)$.]

2. (a) Let S be a set with at least two elements. Define a relation ρ on $\mathcal{P}(S) \setminus \{\varnothing\}$ by declaring that, for $A, B \subseteq S$ with $A, B \neq \varnothing$,

$$A \,\rho\, B \qquad \text{if and only if} \qquad A \cap B \neq \varnothing.$$

Show that ρ is reflexive and symmetric but not transitive.

(b) Let $f : A \to B$ and $g : B \to C$ be functions. Show that if $g \circ f$ is one-to-one then f is one-to-one.

3. (a) Prove that for all integers $m \neq 1$ and for all $n \in \mathbb{N}$,

$$m - 1 \mid m^n - 1$$

(b) Let $S = \{1\} \cup \{\, 1 + \frac{1}{n} : n \in \mathbb{N} \,\}$. Prove that S is not well-ordered.

(c) Let $T = \{1\} \cup \{\, 1 - \frac{1}{n} : n \in \mathbb{N} \,\}$. Prove that T is well-ordered.

4. (a) Find $d = \gcd(1330, 662)$ and find integers u and v such that $d = 1330u + 662v$.

(b) Find a solution to the equation $\overline{662} \cdot_{1330} \overline{x} = \overline{6}$ in $\mathbb{Z}_{1330}$.

(c) Prove that if $f(x)$ is a polynomial with $f(a) = f'(a) = 0$ then $(x - a)^2 \mid f(x)$.

5. (a) Find the least upper bound and the greatest lower bound of the set $S = \{ (\frac{1}{2})^n + (-\frac{1}{3})^n : n \in \mathbb{N} \}$.

(b) Let B be a nonempty subset of $\mathbb{R}$ which is bounded above. Define a second subset A of $\mathbb{R}$ by $A = \{ kb : b \in B \}$ for a fixed $k \in \mathbb{R}$ with $k > 0$. Prove that $\operatorname{lub} A = k \operatorname{lub} B$.

(c) Suppose $\{s_n\}$ and $\{t_n\}$ are real-valued sequences such that $s_n \to s$ and $t_n \to t$ as $n \to \infty$. Suppose further that there exists a fixed integer N such that $s_n < t_n$ for all $n > N$. Is it true that $s < t$? Give a proof or a counterexample.

6. (a) Let $S = (-1, 0] \cup \{1, \frac{1}{2}, \frac{1}{3}, \dots\}$ and define $f : S \to \mathbb{R}$ by $f(x) = \begin{cases} x & x < 0 \\ x + 1 & x \geq 0. \end{cases}$

Prove (from the definition) that f is not continuous at $x = 0$. [Hint: Choose an appropriate ϵ and show that there is no corresponding δ. It might help to draw a picture.]

(b) If $g(x) \to l$ as $x \to a$ prove that $|g(x)| \to |l|$ as $x \to a$. Is the converse true? Give a proof or a counterexample.

(c) Suppose h is continuous on $[0, 1]$, $h(x)$ is *rational* for every $x \in [0, 1]$ and $h(0) = 0$. Find $h(\frac{\sqrt{2}}{2})$. [Hint: Suppose first that $h(\frac{\sqrt{2}}{2}) > 0$ and make use of the Intermediate Value Theorem.]

7. Consider $(\mathbb{Z}, +)$, the group of integers.

(a) Let $S = \{ 4z : z \in \mathbb{Z} \}$. Prove that $(S, +)$ is a subgroup of $(\mathbb{Z}, +)$.

(b) Prove that $\varphi : n \mapsto 4n$ is an isomorphism from $\mathbb{Z}$ to S.

(c) Describe the kernel of φ.

8. (a) Let G be a group and assume that $a^2 = e$ for all $a \in G$ (as usual, e denotes the identity of G). By considering $(ab)^2$, show that G is abelian.

(b) Give an example of a group G with more than 2 elements which has the property that $a^2 = e$ for all $a \in G$.