Defn. Let A be a set. The *identity function* on A is the function $1_A : A \to A$ given by $1_A(x) = x$ for all $x \in A$.

Notice that if $f : A \to B$ is any function then $f = f \circ 1_A$ and $f = 1_B \circ f$.

Defn. Let $f : A \to B$. An *inverse* of f is a function $g : B \to A$ such that

1. For every $x \in A$, $(g \circ f)(x) = x$; and
2. For every $y \in B$, $(f \circ g)(y) = y$.

In other words, g is an inverse of f if $g \circ f = 1_A$ and $f \circ g = 1_B$.

Prop. If a function $f : A \to B$ has an inverse, then the inverse is unique.

Proof 1—standard approach. Suppose g and h are both inverses of f. Then g and h both have domain B. Also, for any $y \in B$ we have $y = (f \circ g)(y)$, so

$$h(y) = h((f \circ g)(y)) = h(f(g(y))),$$

and on the other hand, putting $x = g(y)$ we have $(h \circ f)(x) = x$, in particular with $x = g(y)$ we have

$$h(f(g(y))) = g(y).$$

Putting these together, we get $h(y) = g(y)$. Thus $h = g$. □

Proof 2—slick approach. We know that $h \circ (f \circ g) = (h \circ f) \circ g$, in other words $h \circ 1_B = 1_A \circ g$, so $h = g$. □

So, if f has an inverse, then that inverse is unique. In that case, we denote the inverse of f by f^{-1}. Remember that, before we use f^{-1} as a function, we must show that it exists. We usually do this by appealing to the following theorem.

Theorem 5.2.5 Let $f : A \to B$ be a function. Then f has an inverse if and only if f is a bijection.

Proof. Suppose first that f has an inverse, g say. We must show that f is a bijection, in other words that f is one-to-one and onto.

f **is one-to-one:** Let $x, y \in A$ with $f(x) = f(y)$. Then $g(f(x)) = g(f(y))$, in other words $(g \circ f)(x) = (g \circ f)(y)$, so $x = y$.

f **is onto:** Let $b \in B$. Put $x = g(b)$. Then $x \in A$ and $f(x) = f(g(b)) = (f \circ g)(b) = b$. Hence f is onto.

Thus, if f has an inverse, then f is a bijection.

Conversely, suppose that f is a bijection. [We must show that f has an inverse, in other words we must show that an inverse exists. So we will give a candidate g, show that g really is a function and that g is an inverse of f.] Put

$$g = \{ (y, x) : (x, y) \in f \}.$$

[First we check that g really is a function from B to A, so we check that g is a relation from B to A and every element of B is related to exactly one element of A.] Let $(y, x) \in g$. Then $(x, y) \in f$, so $x \in A$ and $y \in B$. Thus g is a relation from B to A. Now, let $y \in B$. Then, since f is onto, there is some $x \in A$ with $(x, y) \in f$. Then $(y, x) \in g$, so y is related to at least one element of A. Finally, suppose $(y, x) \in g$ and $(y, z) \in g$ [we want to show that $x = z$]. Then $(x, y) \in f$, so $y = f(x)$, and $(z, y) \in f$ so $f(z) = y$. Thus $f(x) = f(z)$, so since f is one-to-one we have $x = z$ as required. Thus $g : B \to A$ is a function.

To finish the proof, we show that g is an inverse of f. Let $x \in A$. Then $(x, f(x)) \in f$, so $(f(x), x) \in g$, so $g(f(x)) = x$. Thus $(g \circ f)(x) = x$ for all $x \in A$, so $g \circ f = 1_A$. Similarly, let $y \in B$. Then $(y, g(y)) \in g$, so $(g(y), y) \in f$, so $f(g(y)) = y$, so $(f \circ g)(y) = y$. Thus $g \circ f = 1_B$.

Hence g is an inverse of f, as required.

□