1. $|f(x) - f(0)| = |x^2(\sin\frac{1}{x} + \cos 5x)| \leq 2x^2$ since $|\sin\frac{1}{x}| \leq 1$ and $|\cos 5x| \leq 1$ for all $x \neq 0$.

So given $\epsilon > 0$ we have $|f(x) - f(0)| < \epsilon$ provided $2x^2 < \epsilon$ i.e. provided $|x| < \sqrt{\epsilon/2}$. Thus if we take $\delta = \sqrt{\epsilon/2}$ then for $|x - 0| < \delta$ we will have $|f(x) - f(0)| < \epsilon$. So f is continuous at 0.

2. (a) For $|x| > 1$ we have $|x^n| > |x|$ for all n and so $\dfrac{1}{|x^n|} < \dfrac{1}{|x|}$. Hence

$$\left|\frac{a_0}{x^n} + \frac{a_1}{x^{n-1}} + \cdots + \frac{a_{n-1}}{x}\right| \leq \left|\frac{a_0}{x^n}\right| + \left|\frac{a_1}{x^{n-1}}\right| + \cdots + \left|\frac{a_{n-1}}{x}\right| < \frac{Kn}{|x|}$$

where $K = \max\{|a_0|, |a_1|, \ldots, |a_{n-1}|\}$. So the LHS is $< \dfrac{1}{2}$ if $\dfrac{Kn}{|x|} < \dfrac{1}{2}$ i.e. if $|x| > 2Kn$ and $|x| > 1$.

(b) We have shown that $-\dfrac{1}{2} \leq \dfrac{a_0}{x^n} + \dfrac{a_1}{x^{n-1}} + \cdots + \dfrac{a_{n-1}}{x} \leq \dfrac{1}{2}$. So

$$\frac{1}{2} \leq 1 + \frac{a_0}{x^n} + \frac{a_1}{x^{n-1}} + \cdots + \frac{a_{n-1}}{x} \leq \frac{3}{2} \quad \text{and} \quad \frac{x^n}{2} \leq x^n\left(1 + \frac{a_0}{x^n} + \frac{a_1}{x^{n-1}} + \cdots + \frac{a_{n-1}}{x}\right) \leq \frac{3x^n}{2}$$

for $|x| > \max\{1, 2Kn\}$. It follows that $x^n + a_{n-1}x^{n-1} + \cdots + a_0$ has the same sign as x^n for large values of $|x|$ i.e. it is positive for $x > 2Kn + 1$ and negative $x < -2Kn - 1$ because n is odd. Hence from Bolzano's Theorem the equation must have a root in the interval $[-2Kn - 1, 2Kn + 1]$.

3. (a) $\left|\dfrac{f(0+h) - f(0)}{h} - 0\right| = \left|\dfrac{f(h)}{h}\right| \leq \dfrac{|h^2|}{|h|} = |h|$. So the LHS is $< \epsilon$ if $|h| < \epsilon$.

So given $\epsilon > 0$ take $\delta = \epsilon$. Then for $|h - 0| = |h| < \delta = \epsilon$ we have $\left|\dfrac{f(0+h) - f(0)}{h} - 0\right| < \epsilon$.
Thus f *is* differentiable at 0 and $f'(0) = 0$.

(b) $f(\sqrt{2}) = 2$. Choose $\epsilon = 1$ (say). Then whatever $\delta > 0$ we choose the interval $(\sqrt{2} - \delta, \sqrt{2} + \delta)$ will always contain rational values of x, i.e. points where $f(x) = 0$. So we will never be able to find an interval of this type for which the values of f lie entirely in the range $(1, 3)$, i.e. in the interval $(f(\sqrt{2}) - \epsilon, f(\sqrt{2}) + \epsilon)$. so f is not continuous at $\sqrt{2}$.

4. Since f''' exists in $[a, b]$ it follows that f'' is continuous in $[a, b]$ and since f'' exists in $[a, b]$ it follows that f' is continuous in $[a, b]$. Similarly f itself must be continuous on $[a, b]$.

So f satisfies the conditions for Rolles Theorem (i.e. it is differentiable in the open interval, continuous in the closed interval and has the same values at the endpoints) and hence there is a point $c \in (a, b)$ such that $f'(c) = 0$. But then f' satisfies the conditions for Rolles Theorem on both the subintervals $[a, c]$ and $[c, b]$. So there are points $q \in (a, c)$ and $r \in (c, b)$ such that $f''(q) = 0$ and $f''(r) = 0$.

Finally f'' satisfies the conditions for Rolles Theorem on the interval $[q, r]$ (or $[r, q]$ if $q > r$). So there is a point $p \in (q, r)$ for which $f'''(p) = 0$.