1. Suppose $a > b$, i.e. $a-b > 0$ and take $\epsilon = 1/n$. Then $0 < a-b < 1/n$ or $n < 1/(a-b)$. But n can be any positive integer. So $1/(b-a)$ is an upper bound for the positive integers. This is a contradiction since we have shown that the positive integers are unbounded above. so the asumption that $a > b$ is false, i.e. $a \leq b$.

2. (a) Recall that if $a > 0$ then $|x| \leq a \Longleftrightarrow -a \leq x \leq a$. Now $-|a| \leq a \leq |a|$ and $-|b| \leq b \leq |b|$. So by addition $-(|a| + |b|) \leq a + b \leq |a| + |b|$. So $|a + b| \leq |a| + |b|$.

(b) $\left|\dfrac{x+3}{3x-2}\right| < 5 \Longleftrightarrow -5 \leq \dfrac{x+3}{3x-2} \leq 5.$

Case 1 $3x - 2 > 0$, i.e. $x > 2/3$. Then $-5(3x - 2) \leq x - 3 \leq 5(3x - 2)$ and so $-15x + 10 \leq x + 3 \leq 15x - 10$ giving $7 \leq 16x$ or $13 \leq 14x$. So $x \geq 7/16, 13/14$ and $2/3$. To satisfy these conditions simultaneously we need $x \geq 13/14$.

Case 2 $3x - 2 < 0$, i.e. $x < 2/3$. Then $-5(3x - 2) \geq x - 3 \geq 5(3x - 2)$ and so $-15x + 10 \geq x + 3 \geq 15x - 10$ giving $7 \geq 16x$ or $13 \geq 14x$. So $x \leq 7/16, 13/14$ and $2/3$. To satisfy these conditions simultaneously we need $x \leq 7/16$.

So the solution set is $(-\infty, 7/16] \cup [13/14, \infty)$.

3. $\dfrac{n-1}{2n} = \dfrac{1}{2} - \dfrac{1}{2n} < \dfrac{1}{2}$ for all $n \in \mathbb{N}$. So $\mathbb{N}$ is bounded above by $1/2$.

This is the least upper bound since if $\epsilon > 0$ then $\dfrac{1}{2} - \dfrac{1}{2n} > \dfrac{1}{2} - \epsilon$ for $\dfrac{1}{2n} < \epsilon$, i.e. for $n > \dfrac{1}{2\epsilon}$. S has no maximum since $\dfrac{1}{2} - \dfrac{1}{2(n+1)}$ is always greater than $\dfrac{1}{2} - \dfrac{1}{2n}$ for any $n \in \mathbb{N}$.

4. $A + B = \{a + b : a \in A, b \in B\}$. If $m = \text{lub}A$ and $n = \text{lub}B$ then $a \leq m$ for all $a \in A$ and $b \leq n$ for all $b \in B$. So $a + b \leq m + n$ for all $a + b \in A + B$, i.e. $m + n$ is an upper bound for $A + B$. Now given $\epsilon > 0$ there are elements $a_0 \in A$ and $b_0 \in B$ such that $a_0 > m - \epsilon/2$ and $b_0 > n - \epsilon/2$ (since m, n are least upper bounds for A, B respectively). Then $a_0 + b_0 \in A + B$ and $a_0 + b_0 > m + n - \epsilon$. So $m + n$ is the lub for $A + B$.

If $A \subset B$ then clearly n is an upper bound for A. So (by the definition of a least upper bound) $\text{lub}A \leq n$.

5. Firstly $a \leq x$ for all $a \in A$. So x is an upper bound for A. But $x \in A$ and $x > x - \epsilon$ for any $\epsilon > 0$. So $x - \epsilon$ is not an upper bound. Hence x is the least upper bound for A.

6. $y - x > 0$ and since the positive integers are unbounded above there is an integer $n > \sqrt{2}/(y - x)$, i.e. $0 < \sqrt{2}/n < y - x$. Now $\sqrt{2}/n$ is irrational since if $\sqrt{2}/n = p/q$ then $\sqrt{2} = np/q$, etc.
Let m be the smallest multiple of $\sqrt{2}/n$ which is $\geq x$. (Such multiples exist by the Archimedean property and there is a smallest by the well-ordering property of the positive integers.) If $m\sqrt{2}/n =$

x take $s = (m+1)\sqrt{2}/n$. Then s is irrational and clearly $x < s < y$. If $m\sqrt{2}/n > x$ then $(m-1)\sqrt{2}/n < x$ and we have $x < \dfrac{(m-1)+1}{n}\sqrt{2} < x + \dfrac{\sqrt{2}}{n} < y$. Now take $s = m\sqrt{2}/n$.