1. $a = \prod_{i=1}^{n} p_i^{a_i}$, $b = \prod_{i=1}^{n} p_i^{b_i}$, $c = \prod_{i=1}^{n} p_i^{c_i}$, $a_i, b_i, c_i \geq 0$.

$\operatorname{lcm}(ca, cb) = \prod_{i=1}^{n} p_i^{r_i}$, where $r_i = \max\{a_i + c_i, b_i + c_i\} = c_i + \max\{a_i, b_i\}$.

So $\operatorname{lcm}(ca, cb) = \prod_{i=1}^{n} p_i^{c_i} p_i^{\max\{a_i, b_i\}} = c \operatorname{lcm}(a, b)$.

2. (a) $d|n \Leftrightarrow d = \prod_{i=1}^{r} p_i^{d_i}$ with $0 \leq d_i \leq e_i$.

The number of such d is the number of possible ways of choosing the exponents d_i. For each prime p_i, there are $e_i + 1$ choices for d_i, namely $d_i = 0, 1, \cdots, e_i$. So the number of possibilities is $(e_1 + 1)(e_2 + 1) \cdots (e_r + 1)$.

(b) Suppose $n = ab$. Then $a = \prod p_i^{a_i}$, $b = \prod p_i^{b_i}$ with $0 \leq a_i, b_i$ and $a_i + b_i = e_i$. a and b are relatively prime if and only if $a_i = 0$ or e_i (which means that $b_i = e_i$ or 0 respectively) for each i. In other words for $i = 1, \cdots, r$ there are two possibilities for each exponent a_i. That means there are 2^r possible values for a. Since $ab = ba$, the number of distinct factorizations in this form is $\frac{2^r}{2} = 2^{r-1}$.

3. (a) $65 \equiv -1 \bmod 11$ so
$65^{67} \equiv (-1)^{67} \bmod 11$
$\equiv -1 \bmod 11$
$\equiv 10 \bmod 11$, the remainder is 10.

(b) $13 \equiv -2 \bmod 5$
$13^2 \equiv 4 \bmod 5$
$\equiv -1 \bmod 5$
$13^4 \equiv 1 \bmod 5$.
$189 = 4 \cdot 47 + 1$ so
$13^{189} \equiv (13^4)^{47} \cdot 13$
$\equiv 1^{47} \cdot (-2) \bmod 5$
$\equiv -2 \bmod 5$
$\equiv 3 \bmod 5$, the remainder is 3.

4. (a) (i)

$\bar{x}$	$\bar{0}$	$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{4}$	$\bar{5}$	$\bar{6}$	$\bar{7}$
$\bar{4} \cdot_8 \bar{x}$	$\bar{0}$	$\bar{4}$	$\bar{0}$	$\bar{4}$	$\bar{0}$	$\bar{4}$	$\bar{0}$	$\bar{4}$

So no solutions to $\bar{6} = \bar{4} \cdot_8 \bar{x}$.

(ii)

$\bar{x}$	$\bar{0}$	$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{4}$	$\bar{5}$	$\bar{6}$	$\bar{7}$	$\bar{8}$	$\bar{9}$
$\bar{4} \cdot_{10} \bar{x}$	$\bar{0}$	$\bar{4}$	$\bar{8}$	$\bar{2}$	$\bar{6}$	$\bar{0}$	$\bar{4}$	$\bar{8}$	$\bar{2}$	$\bar{6}$

Solutions are $\bar{x} = \bar{4}, \bar{9}$.

(iii)

$\bar{x}$	$\bar{0}$	$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{4}$	$\bar{5}$	$\bar{6}$	$\bar{7}$	$\bar{8}$	$\bar{9}$	$\bar{10}$	$\bar{11}$	$\bar{12}$	$\bar{13}$	$\bar{14}$
$\bar{4} \cdot_{15} \bar{x}$	$\bar{0}$	$\bar{4}$	$\bar{8}$	$\bar{5}$	$\bar{1}$	$\bar{4}$	$\bar{9}$	$\bar{13}$	$\bar{2}$	$\bar{6}$	$\bar{10}$	$\bar{14}$	$\bar{3}$	$\bar{7}$	$\bar{11}$

Unique solution $\bar{x} = \bar{9}$. (We might have expected this since $\bar{4}$ is invertible is $\mathbb{Z}_{15}$ because $\gcd(4, 15) = 1$).

(b) $\overline{143}$ is invertible is $\mathbb{Z}_{368} \Leftrightarrow \gcd(143, 368) = 1$. We use the Euclidean algorithm to find $d = \gcd(143, 368)$ and to find u, v such that $d = 143u + 368v$.

368	1	0
143	0	1
82	1	-2
61	-1	3
21	2	-5
19	-5	13
2	7	-18
1	-68	175.

So $\gcd(143, 368) = 1$ and $1 = (368)(-68) + (143)(175)$.
Hence $\overline{143}$ is invertible and $(\overline{143})^{-1} = \overline{175}$).

5. $b(x) = a(x)(x-2) + (5x^3 - 4x^2 + 10x - 8) = aq_1 + r_1$
$a(x) = r_1(x)(\frac{1}{5}x + \frac{14}{25}) + (-\frac{19}{25}x^2 - \frac{38}{25}) = r_1q_2 + r_2$
$r_1(x) = r_2(x)(-\frac{125}{19}x + \frac{100}{19})$, remainder 0.
So $\gcd(a(x), b(x)) = r_2(x) = -\frac{19}{25}x^2 - \frac{38}{25}$ or to simplify, could use $-\frac{25}{19}r_2(x) = x^2 + 2$.
$r_2 = a - r_1q_2 = a - (b - aq_1)q_2 = a(1 + q_1q_2) + b(-q_2) = au + bv$.
$u = 1 + q_1q_2 = \frac{1}{5}x^2 + \frac{4}{25}x - \frac{3}{25}$.
$v = -q_2 = -\frac{1}{5}x - \frac{14}{25}$.
Alternatively,
$x^2 + 2 = a(x)(-\frac{5}{19}x^2 - \frac{4}{19}x + \frac{3}{19}) + b(x)(\frac{5}{19}x + \frac{14}{19})$.