1. Prove that for all $n \in \mathbb{N}$,

$$3 \mid 2^{2^n} - 1.$$

Proof: [Induction on n.] P_n: $3 \mid 2^{2^n} - 1$

P_1 is true:

Proof: $2^{2^1} - 1 = 4 - 1 = 3$ and $3 \mid 3$.

For all $k \geq 1$ $P_k \Rightarrow P_{k+1}$ is true.

Proof: Assume $k \geq 1$ and that $3 \mid 2^{2^k} - 1$, so that $2^{2^k} - 1 = 3x$ for some $x \in \mathbb{Z}$.

$$\begin{aligned}
2^{2^{k+1}} - 1 &= 2^{2(2^k)} - 1 = (2^{2^k})^2 - 1^2 \\
&= (2^{2^k} + 1)(2^{2^k} - 1) \\
&= (2^{2^k} + 1)(3x) \text{ by induction hypothesis} \\
&= 3x(2^{2^k} + 1). \text{ Hence } 3 \mid 2^{2^{k+1}} - 1.
\end{aligned}$$

By PMI, P_n is true for all $n \in \mathbb{N}$.

2. Prove that if r, s are relatively prime integers, then so are $r + s$ and s.

Proof: Suppose $0 < a \in \mathbb{Z}$ and $a \mid r + s$ and $a \mid s$. We show $a = 1$.

$r + s = ax$ and $s = ay$ for some $x, y, \in \mathbb{Z}$. So $r = ax - s = ax - ay = a(x - y)$. So $a \mid r$ (and of course $a \mid s$ still).

a is a common divisor of r, s which are relatively prime so $a \mid 1$, so that $a = 1$. Hence $r + s$ and s are relatively prime.

Alternatively, we could use the fact that a, b are relatively prime if and only if 1 is a linear combination of a, b:

$rx + sy = 1$ for some $x, y \in \mathbb{Z}$. So $rx + sy + sx - sx = 1$. Hence $(r + s)x + s(y - x) = 1$. So $r + s, s$ are relatively prime.

3. Suppose that a, b are non-zero integers and that m is a positive common multiple of a, b. Prove that the following statements are equivalent:

(i) $m \leq c$ for every positive common multiple c of a, b.

(ii) $m \mid c$ for all common multiples c of a, b.

Proof: $m > 0$ and $a \mid m$ and $b \mid m$.

Assume (i), that is, assume $m \leq c$ for all positive common multiples c of a, b.

Let c be a common multiple of a, b. We show $m \mid c$. We may assume $c > 0$ since $m \mid c$ if and only if $m \mid -c$.

By the Division Algorithm,

$$c = mq + r \quad \text{for some } q, r \in \mathbb{Z}, \quad 0 \leq r < m.$$

Suppose $r \neq 0$. Since c and m are common multiples of a, b, then so is $c - mq = r$. But then from (i), $m \leq r$, a contradiction. Hence $r = 0$ and $m \mid c$. (i) $\Rightarrow$ (ii) is proved.

Assume (ii), that is $m \mid c$ for all common multiples c of a, b. If c is positive and $m \mid c$, then $m \leq c$. Hence (ii) $\Rightarrow$ (i) is proved.

4. Prove that if a, b are relatively prime integers and $a \mid c$ and $b \mid c$ then $ab \mid c$.

Proof: a, b relatively prime means $ax + by = 1$ for some integers x, y. Suppose $a \mid c$ and $b \mid c$. Then $au = c$ and $bv = c$ for some integers u, v.

$$\begin{aligned} ax + by = 1 &\Rightarrow axc + byc = c \\ &\Rightarrow axbv + byau = c \\ &\Rightarrow ab(xv + yu) = c \\ &\Rightarrow ab \mid c \end{aligned}$$