1. $P_n : \sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1}$.

P_1 is true.

Proof: $\frac{1}{1(2)} = \frac{1}{1+1}$.

For all $k \geq 1, P_k \Rightarrow P_{k+1}$ is true.

Proof: Suppose $k \geq 1$ and P_k is true (inductive hypothesis).

$$\begin{aligned} & \sum_{i=1}^{k+1} \frac{1}{i(i+1)} = \sum_{i=1}^{k} \frac{1}{i(i+1)} + \frac{1}{(k+1)(k+2)} \\ = \; & \frac{k}{k+1} + \frac{1}{(k+1)(k+2)} \qquad \text{(using inductive hypothesis)} \\ = \; & \frac{1}{k+1}(k + \frac{1}{k+2}) = \frac{1}{k+1}\left(\frac{k^2+2k+1}{k+2}\right) \\ = \; & \frac{(k+1)^2}{(k+1)(k+2)} = \frac{k+1}{k+2}. \end{aligned}$$

So P_{k+1} is true.

By the Principle of Mathematical Induction, P_n is true for all $n \geq 1$.

2. $P_n : \sum_{i=1}^{n} \frac{i}{2^i} = 2 - \frac{n+2}{2^n}$.

P_1 is true.

Proof: $\frac{1}{2^1} = 2 - \frac{1+2}{2} = \frac{1}{2}$.

For all $k \geq 1, P_k \Rightarrow P_{k+1}$ is true.

Proof: Suppose $k \geq 1$ and P_k is true (inductive hypothesis).

$$\begin{aligned} & \sum_{i=1}^{k+1} \frac{i}{2^i} = \sum_{i=1}^{k} \frac{i}{2i} + \frac{k+1}{2^{k+1}} \\ = \; & 2 - \frac{k+2}{2^k} + \frac{k+1}{2^{k+1}} \text{(using inductive hypothesis)} \\ = \; & 2 - \frac{2(k+2)-(k+1)}{2^{k+1}} \\ = \; & 2 - \frac{(k+1)+2}{2^{k+1}}. \end{aligned}$$

So P_{k+1} is true.

By the Principle of Mathematical Induction, P_n is true for all $n \geq 1$.

3. P_n: For all $m \in \mathbb{N}$ if A has n elements and B has m elements, then there are m^n functions $f : A \to B$.

P_1 is true.

Proof: Let $A = \{a\}$, a set with one element. $B = \{b_1, \dots, b_m\}$ a set with m elements. A function $f : A \to B$ is completely determined by the value $f(a)$, and there are m possible such values, namely $b_1, \dots, b_m$.

For all $k \geq 1$ $P_k \Rightarrow P_{k+1}$ is true.

Proof: Assume $k \geq 1$ and that P_k is true, and assume that A is a set with $k+1$ elements and that $B = \{b_1, \dots, b_m\}$ has m elements.

Let $a \in A$. ($A \neq \emptyset$ since $k + 1 > 0$).

$A - \{a\}$ has k elements, so by the inductive hypothesis there are exactly k^m functions
$f : A - \{a\} \to B$.

For each such function f there are m functions $f_1, \dots, f_m : A \to B$ defined by

$$f_i(x) = \begin{cases} f(x) & \text{if } x \neq a \\ b_i & \text{if } x = a. \end{cases}$$

Hence there are at least $m(m^k) = m^{k+1}$ functions $f : A \to B$. But obviously every function from $A \to B$ is one of these. So the number is exactly m^{k+1}.

By PMI, P_n is true for all $n \in \mathbb{N}$.

4. P_n: In any pile of n coins where it is known that one is heavier than the rest and all the others are the same weight, four weighings are sufficient to discover the odd one.

P_2 is true.

Proof: Put one coin on each side of the balance. The heavy one is discovered in 1 weighing.

For all $k \geq 2$, $P_k \Rightarrow P_{k+1}$ is true.

"Proof": Suppose $k \geq 2$ and that P_k is true and that a pile of $k+1$ coins is given. Select one coin and set it aside. By the induction hypothesis, 4 weighings are enough to find the heavy coin among the remaining k coins if it is there. If it is not there, then it is the coin previously set aside. In any case, it is discovered by 4 weighings.

Hence, by PMI, P_n is true for all $n \geq 2$.

But wait. Surely something is wrong here, because clearly one can't sort arbitrarily large piles of coins with only 4 weighings. In fact, this same argument would work equally well if the number of weighings were only 1 instead of 4. Look again at the statement

P_n: In any pile of n coins where it is known that one is heavier than the rest and all the others are the same weight, four weighings are sufficient to discover the odd one.

The induction argument above fails because when one coin is removed from the pile of $k+1$, the condition that one is heavier than the rest is no longer certainly satisfied, so the inductive hypothesis P_k does not apply.
