1. (a) (i) $(1,1) \sqsubseteq (2,3)$.
(ii) $(1,-1) \not\sqsubseteq (2,-2)$.
(iii) $(1,2) \not\sqsubseteq (2,1)$.
(iv) $(2,1) \not\sqsubseteq (1,2)$.

(b) We must chekc that $\sqsubseteq$ is reflexive, antisymmetric and transitive.

Reflexive: Let $(x,y) \in \mathbb{R}^2$. Then $x \le x$ and $y \le y$, so $(x,y) \sqsubseteq (x,y)$.

Antisymmetric: Let $(x,y),(u,v) \in \mathbb{R}^2$ with $(x,y) \sqsubseteq (u,v)$ and $(u,v) \sqsubseteq (x,y)$. Then $x \le u$ and $y \le v$, and $u \le x$ and $v \le y$. Since $x \le u \le x$ we have $x = u$, and since $y \le v \le y$ we have $y = v$. So $(x,y) = (u,v)$.

Transitive: $(x,y),(u,v),(s,t) \in \mathbb{R}^2$ with $(x,y) \sqsubseteq (u,v)$ and $(u,v) \sqsubseteq (s,t)$. Then $x \le u$ and $y \le v$, and $u \le s$ and $v \le t$. So $x \le u \le s$ and $y \le v \le t$, so $x \le s$ and $y \le t$, so $(x,y) \sqsubseteq (s,t)$.

(c) $\sqsubseteq$ is not a total order on $\mathbb{R}^2$: from (a)(iii) and (iv) we see that $(1,2)$ and $(2,1)$ are not comparable under $\sqsubseteq$.

2. We must show that $\sim_f$ is reflexive, symmetric and transitive.

Reflexive: Let $x \in A$. Then $f(x) = f(x)$, so $x \sim_f x$.

Symmetric: Let $x, y \in A$ with $x \sim_f y$. Then $f(x) = f(y)$, so $f(y) = f(x)$, so $y \sim_f x$.

Transitive: Let $x, y, z \in A$ with $x \sim_f y$ and $y \sim_f z$. Then $f(x) = f(y)$, and $f(y) = f(z)$, so $f(x) = f(z)$, so $x \sim_f z$.

3. (a) We must show that $\sim$ is reflexive, symmetric and transitive.

Reflexive: Let $(x,y) \in \mathbb{R}^2$. Then $3x - y = 3x - y$, so $(x,y) \sim (x,y)$.

Symmetric: Let $(u,v),(x,y) \in \mathbb{R}^2$ with $(u,v) \sim (x,y)$. Then $3u-v = 3x-y$, so $3x-y = 3u-v$, so $(x,y) \sim (u,v)$.

Transitive: Let $(u,v),(x,y),(z,w) \in \mathbb{R}^2$ with $(u,v) \sim (x,y)$ and $(x,y) \sim (z,w)$. Then $3u-v = 3x-y$ and $3x-y = 3z-w$, so $3u-v = 3z-w$, so $(u,v) \sim (z,w)$.

(b) For all $(x,y) \in \mathbb{R}^2$ we have

$$(x,y) \in T_{(0,0)} \iff (0,0) \sim (x,y) \iff 3 \cdot 0 - 0 = 3x - y \iff y = 3x.$$

Thus $T_{(0,0)}$ is the line $y = 3x$ with slope 3, passing through the origin.

(c) Similarly, for all $(x,y) \in \mathbb{R}^2$ we have

$$(x,y) \in T_{(u,v)} \iff (u,v) \sim (x,y) \iff 3u-v = 3x-y \iff y = 3x+(v-3u) \iff y = 3x+c,$$

where $c = v - 3u$. Thus $T_{(u,v)}$ is the line $y = 3x + (v-3u)$ with slope 3 and y-intercept $v - 3u$.

(d) The set $\mathcal{R}_\sim$ of equivalence classes under $\sim$ is the set of all lines with slope 3.

4. Suppose that $g \circ f$ is onto and g is one-to-one. Let $b \in B$. [We want to use the fact that $g \circ f$ is onto, and to do that we need to get an element of C. We can get one by applying g to b.] Then $g(b) \in C$, and $g \circ f$ is onto, so there is some $x \in A$ with $(g \circ f)(x) = g(b)$, in other words $g(f(x)) = g(b)$. Then, since g is one-to-one, we have $f(x) = b$, as required. Hence f is onto.