1. (1) $\implies$ (2)**:** Suppose $A \subseteq B$. Let $x \in A \cap B$. Then $x \in A$ and $x \in B$. In particular, $x \in A$. Thus $A \cap B \subseteq A$. Conversely, let $y \in A$. Since $A \subseteq B$, we also have $y \in B$, so $y \in A \cap B$. Thus $A \subseteq A \cap B$. Combining these, we have $A \cap B = A$.

(2) $\implies$ (3)**:** Suppose $A \cap B = A$. Suppose, for a contradiction, that $A \setminus B \neq \varnothing$. Let $x \in A \setminus B$. Then $x \in A$ and $x \notin B$. Since $x \in A = A \cap B$, we have $x \in A$ and $x \in B$. But this contradicts the earlier assertion that $x \notin B$. So there is no such x, i.e. $A \setminus B = \varnothing$.

(3) $\implies$ (1)**:** We'll prove the contrapositive. Suppose that $A \setminus B \neq \varnothing$. Then there is some $x \in A \setminus B$. Then $x \in A$ and $x \notin B$, so it is not true that every element of A is an element of B. Thus $A \nsubseteq B$.

2. Suppose that $A \subseteq B$. Let $x \in \mathcal{P}(A)$. Then $x \subseteq A \subseteq B$, so $x \subseteq B$, so $x \in \mathcal{P}(B)$. Thus $\mathcal{P}(A) \subseteq \mathcal{P}(B)$.

Conversely, suppose that $\mathcal{P}(A) \subseteq \mathcal{P}(B)$. Since $A \subseteq A$, we have $A \in \mathcal{P}(A) \subseteq \mathcal{P}(B)$. Thus $A \in \mathcal{P}(B)$, so $A \subseteq B$.

3. (a) $\mathcal{P}(A) = \{\varnothing, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$.

(b) $\mathcal{P}(B) = \{\varnothing, \{1\}, \{4\}, \{1,4\}\}$.

(c) $\mathcal{P}(A \cap B) = \mathcal{P}(\{1\}) = \{\varnothing, \{1\}\}$.

(d) $\mathcal{P}(A \cup B) = \mathcal{P}(\{1,2,3,4\})$, so

$$\begin{aligned}\mathcal{P}(A \cup B) = \{\varnothing, \{1\}, \{2\}, \{3\}, \{4\},\\ \{1,2\}, \{1,3\}, \{1,4\}, \{2,3\}, \{2,4\}, \{3,4\},\\ \{1,2,3\}, \{1,2,4\}, \{1,3,4\}, \{2,3,4\}, \{1,2,3,4\}\}.\end{aligned}$$

4. (a) Let A and B be sets. Let $x \in \mathcal{P}(A \cap B)$. Then $x \subseteq A \cap B$. For any $a \in x$ we have $a \in A \cap B$, so $a \in A$ and $a \in B$. Thus every element of x is an element of A, so $x \subseteq A$, and every element of x is an element of B, so $x \subseteq B$. So $x \in \mathcal{P}(A)$ and $x \in \mathcal{P}(B)$, so $x \in \mathcal{P}(A) \cap \mathcal{P}(B)$. Hence $\mathcal{P}(A \cap B) \subseteq \mathcal{P}(A) \cap \mathcal{P}(B)$.

Conversely, let $y \in \mathcal{P}(A) \cap \mathcal{P}(B)$. Then $y \in \mathcal{P}(A)$ and $y \in \mathcal{P}(B)$, so $y \subseteq A$ and $y \subseteq B$. Let $a \in y$. Then $a \in A$ (since $y \subseteq A$) and $a \in B$ (since $y \subseteq B$), so $a \in A \cap B$. Thus $y \subseteq A \cap B$, so $y \in \mathcal{P}(A \cap B)$. Hence $\mathcal{P}(A) \cap \mathcal{P}(B) \subseteq \mathcal{P}(A \cap B)$.

Combining these, we get $\mathcal{P}(A \cap B) = \mathcal{P}(A) \cap \mathcal{P}(B)$.

(b) Let $A = \{1,2,3\}$ and let $B = \{1,4\}$. From Question 3(d) we know that $\{1,2,3,4\} \in \mathcal{P}(A \cup B)$. However, from 3(a) we know that $\{1,2,3,4\} \notin \mathcal{P}(A)$, and from 3(b) we know that $\{1,2,3,4\} \notin \mathcal{P}(B)$. Thus $\{1,2,3,4\} \notin \mathcal{P}(A) \cup \mathcal{P}(B)$, so in this example we have $\mathcal{P}(A \cup B) \neq \mathcal{P}(A) \cup \mathcal{P}(B)$.