1. No, $(G, \cdot)$ is not a group. For example, the operation is not associative:

$$\begin{aligned}(a \cdot b) \cdot c &= (5a - 4b) \cdot c \\ &= 5(5a - 4b) - 4c \\ &= 25a - 20b - 4c\end{aligned} \qquad \begin{aligned}a \cdot (b \cdot c) &= a \cdot (5b - 4c) \\ &= 5a - 4(5b - 4c) \\ &= 5a - 20b + 16c \\ &\neq (a \cdot b) \cdot c\end{aligned}$$

Take for example $a = 1$, $b = -1$, $c = 2$, then $(a \cdot b) \cdot c \neq a \cdot (b \cdot c)$.

Neither does $(G, \cdot)$ have an identity.

2. We have the following Cayley table for G:

$\cdot_{18}$	1	5	7	11	13	17
1	1	5	7	11	13	17
5	5	7	17	1	11	13
7	7	17	13	5	1	11
11	11	1	5	13	17	7
13	13	11	1	17	7	5
17	17	13	11	7	5	1

We can see that the group is closed and has an identity 1, from the Cayley table. We can also see that each element has an inverse: $5^{-1} = 11$, $7^{-1} = 13$, $11^{-1} = 5$, $13^{-1} = 7$ and 17 is its own inverse. As we are using multiplication in $\mathbb{R}$ we also know that the operation is associative.

3. Assume that $(a * b)^2 = a^2 * b^2$ for all $a, b \in G$. Then

$$(a * b) * (a * b) = a * a * b * b.$$

Hence

$$a^{-1} * a * b * a * b * b^{-1} = a^{-1} * a * a * b * b * b^{-1}.$$

It follows that $b * a = a * b$. Therefore G is abelian.

4. Recall that for an invertible 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, the inverse is given by

$$A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix},$$

where $det(A) = ad - bc$. Hence a matrix with determinant zero will not have an inverse, thus S does not form a group under the operation of matrix multiplication.

5. (a) We must make sure that we stay in S when calculating $a * b$ for $a, b \in S$. In other words, we want to be sure that if neither a nor b is -1 then $a * b$ does not equal -1 either. It is clear from the definition that $a * b \in \mathbb{R}$. The question is, will $a * b \in S$? So assume $a, b \in S$ and $a * b = -1$. Then

$$a * b = a + b + ab = -1 \qquad \Rightarrow \qquad a(1+b) = -1 - b \qquad \Rightarrow \qquad a(1+b) = -(1+b).$$

Therefore, if $b \neq -1$ (which must be the case, since $b \in S$) we can divide by $1+b$ to get $a = -1$ which contradicts the assumption that $a \in S$. Result: if $a \neq -1$ and $b \neq -1$ then $a * b \neq -1$, or in other words if $a, b \in S$ then $a * b \in S$, i.e. S is closed under the operation $*$.

(b) We have established closure, so it remains to establish the three defining properties of a group:

Associativity: Is it true that $a * (b * c) = (a * b) * c$? We compute the expression on the left and the expression on the right separately, and then look if the results are equal.

$$\begin{aligned} a * (b * c) &= a * (b + c + bc) \\ &= a + (b + c + bc) + a(b + c + bc) \\ &= a + b + c + ab + ac + abc, \end{aligned}$$

and

$$\begin{aligned} (a * b) * c &= (a * b) + c + (a * b)c \\ &= (a + b + ab) + c + (a + b + ab)c \\ &= a + b + c + ab + ac + abc. \end{aligned}$$

Identity: We want to find out if there exists an element e in S such that $e * a = a * e = a$ for all $a \in S$. We note first of all that it is clear from the definition that $a * b = b * a$, hence we need only concern ourselves with finding an e such that $e * a = a$. The element e must satisfy

$$e * a = e + a + ea = a \qquad \text{for all } a \in S.$$

Now one sees immediately that this condition is satisfied if we use $e = 0$. (If you don't see it immediately, solve the equation $e(1 + a) = 0$, bearing in mind that $a \neq -1$.)

Inverses: Given any element $a \in S$, does there exist an element b, also in S, such that $a*b = 0$ (the identity element which we have just found)? Let's see:

$$a * b = a + b + ab = 0.$$

We try to solve for b: $b(1+a) = -a$, and hence $b = -\dfrac{a}{1+a}$. Again everything works out well because $a \neq -1$. So, if there is an element b with the required property, it can only be $b = -\dfrac{a}{1+a}$. But it is a simple calculation to check that

$$a * \left(-\frac{a}{1+a}\right) = a + \frac{-a}{1+a} + \frac{-a^2}{1+a} = \frac{a + a^2 - a - a^2}{1+a} = 0.$$