Continuity

Defn By a *neighbourhood* of a we mean an open interval containing a. In particular we have the ϵ-nbd $B(a, \epsilon) = \{x : |x - a| < \epsilon\}$. this is often called the ϵ-ball about a.

Defn Let f be a function whose domain and range are in $\mathbb{R}$ and suppose that $a \in D(f)$ (the domain of f). Then f is *continuous* at a if for any given nbd V of $f(a)$ there exists a nbd U of a such that $f(U) \subset V$, i.e. $f(x) \in V$ whenever $x \in U$.

Remarks
1. We wish to show that the values of the function are within a prescribed distance of the value $f(a)$ (given by V) whenever x is "close enough" to a. So V is always pre-determined and the task is to find a suitable nbd U of a (if possible) which is mapped by f entirely into V. Then U expresses the notion of being *close enough to a*.

2. Clearly ever nbd of a contains an ϵ-nbd of a. So it is enough to have the result for ϵ-nbds. This leads back to the more classical definition that f is continuous at a if, given $\epsilon > 0$, there exists a $\delta > 0$ (depending in general on both ϵ and a) such that

$$|f(x) - f(a)| < \epsilon \qquad \text{whenever} \qquad |x - a| < \delta.$$

Theorem If f and g are both continuous at a then so are $f + g$ and fg.

Proof Given $\epsilon > 0$ there exist $\delta_1, \delta_2 > 0$ such that

$$|f(x) - f(a)| < \epsilon/2, \quad |g(x) - g(a)| < \epsilon/2$$

whenever $|x - a| < \delta_1, \delta_2$ respectively. Let $\delta = \min(\delta_1, \delta_2)$. Then for $|x - a| < \delta$ we have

$$|(f + g)(x) - (f + g)(a)| \le |f(x) - f(a)| + |g(x) - g(a)| \le \epsilon/2 + \epsilon/2 = \epsilon$$

So $f + g$ is continuous at a.

For the continuity of fg without loss of generality let $\epsilon < 1$ and put $M = \max(|f(a)|, |g(a)|, 1)$. Then there exists $\delta > 0$ such that $|f(x) - f(a)| < \epsilon/3M$, $|g(x) - g(a)| < \epsilon/3M$ for $|x - a| < \delta$. Furthermore $|f(x)| < |f(a)| + \epsilon/3M \le M + \epsilon/3M$. Now

$$\begin{aligned} |(fg)(x) - (fg)(a)| &= |f(x)g(x) - f(x)g(a) + f(x)g(a) - f(a)g(a)| \\ &\le |f(x)||g(x) - g(a)| + |g(a)||f(x) - f(a)| \\ &< \left(M + \frac{\epsilon}{3M}\right)\frac{\epsilon}{3M} + M\frac{\epsilon}{3M} \\ &< \frac{\epsilon}{3} + \frac{\epsilon^2}{9M^2} + \frac{\epsilon}{3} \;<\; \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} \;<\; \epsilon \end{aligned}$$

So fg is continuous at a.

Theorem If g is continuous at a and $g(a) \neq 0$ then $1/g$ is continuous at a.

Proof Take $\epsilon = \frac{1}{2}|g(a)|$. Since g is continuous at a there exists $\delta_1 > 0$ such that $|g(x) - g(a)| < \epsilon$ if $|x - a| < \delta_1$, i.e. if $|x - a| < \delta_1$ then

$$\begin{aligned} |g(x)| &= |g(x) - g(a) + g(a)| \leq |g(x) - g(a)| + |g(a)| < \epsilon + |g(a)| = \tfrac{3}{2}|g(a)|, \quad \text{and} \\ |g(a)| &= |g(a) - g(x) + g(x)| \leq |g(a) - g(x)| + |g(x)| < \epsilon + |g(x)| = \tfrac{1}{2}|g(a)| + |g(x)| \end{aligned}$$

i.e. $\frac{1}{2}|g(a)| < |g(x)|$, and so $\frac{1}{2}|g(a)| < |g(x)| < \frac{3}{2}|g(a)|$.

Now let ϵ be arbitrary. Again since g is continuous at a there exists $\delta > 0$ (which can certainly be taken $\leq \delta_1$) such that $|g(x) - g(a)| < \epsilon_1 = \dfrac{\epsilon |g(a)|^2}{2}$ when $|x - a| < \delta$ $(\leq \delta_1)$. But then

$$\left| \frac{1}{g(x)} - \frac{1}{g(a)} \right| = \left| \frac{g(x) - g(a)}{g(x)g(a)} \right| \leq \frac{|g(x) - g(a)|}{\frac{1}{2}|g(a)|^2} < \frac{\epsilon_1}{|g(a)|^2} = \epsilon$$

Hence $1/g$ is continuous at a.

Remarks

1. Note that the first part of the proof is devoted to showing that $|g(x)|$ is bounded below by a *positive* number when x is close to a. This estimate is needed because $g(x)$ occurs in the denominator in the second part of the proof.

2. Combining this result with that concerning products we see that f/g $(= f \cdot 1/g)$ is continuous at a when both f and g are and $g(a) \neq 0$. In particular rational functions are continuous at all points where the denominator is zero.

Theorem (Composite functions) Assume that f is continuous at a and g is continuous at $b = f(a)$. then the composite function $h = g \circ f$ is continuous at a.

Proof Since g is continuous at b, for every nbd W of $g(b)$ there is a nbd V of b such that $g(V) \subset W$. But $b = f(a)$ and f is continuous at a, so for the nbd V there is another nbd U such that $f(U) \subset V$. Now $h(a) = g \circ f(a) = g(b)$ and clearly $h(U) \subset W$ i.e. for every nbd W of $h(a)$ there is a nbd U of a such that $h(U) \subset W$. Hence $h = g \circ f$ is continuous at a.

NB The proof can easily be recast in ϵ, δ terms.

Types of discontinuities Discontinuities of a function $f(x)$ can be classified as "removeable", "jump", "infinite" and "essential". In the first case if the missing value is l then we see that $f(x)$ approaches l as x gets close (from either side) to the point a at which the discontinuity occurs. We write $f(x) \to l$ as $x \to a$, or $\lim_{x\to a} f(x) = l$, and call l the limit of f as x tends to a.

Defn $\lim_{x\to a} f(x) = l$ iff given $\epsilon > 0$ there is a $\delta > 0$ (depending in general on both ϵ and a) such that

$$|f(x) - l| < \epsilon \qquad \text{whenever} \qquad 0 < |x - a| < \delta.$$

Remarks

1. Similar definitions hold for one-sided limits.

2. The definition is almost the same as that for the continuity of f at a except that l is not necessarily $f(a)$ and we consider only the *deleted* nbd $B(a, \delta) - \{a\}$ for a i.e. $\{x : 0 < |x - a| < \delta\}$. So the following result can be taken as a definition of continuity (at the point a).

Theorem f is continuous at a if:
(i) f is defined at a.
(ii) $f(x) \to f(a)$ as $x \to a$.

3. It should be noted that continuity is a property of a function at individual points. We can find examples of functions that are continuous at only one point or only at the irrational numbers.

4. Just as for sequences limits are unique, standard results about the limits of sums, products and quotients hold and we have the Sandwich theorem.

5. Similar results hold concerning limits as $x \to \pm\infty$. In these cases the condition $0 < |x - a| < \delta$ is replaced by conditions such as $x > K(> 0)$.

Three important results

The following results concern the behaviour of continuous functions on a closed interval and are important for the development of calculus.

Theorem (Bolzano) If f is continuous on $[a, b]$ and $f(a) < 0 < f(b)$, then there is some c in $[a, b]$ such that $f(c) = 0$.

Cor (Intermediate Value Theorem) If f is continuous on $[a, b]$ and k is any number between $f(a)$ and $f(b)$ then there is at least one number c netween a and b such that $f(c) = k$.

Theorem (Boundedness Theorem) If f is continuous on $[a, b]$ then f is bounded on $[a, b]$, i.e. there exists M such that $|f(x)| \leq M$ for all x in $[a, b]$.

Theorem (Extreme Value Theorem) If f is continuous on $[a, b]$, then there are numbers c, d in $[a, b]$ such that $f(d) \leq f(x) \leq f(c)$ for all x in $[a, b]$. Furthermore

$$\begin{aligned} f(c) &= \max\{f(x) : x \in [a, b]\} = \text{lub}\{f(x) : x \in [a, b]\} \\ f(d) &= \min\{f(x) : x \in [a, b]\} = \text{glb}\{f(x) : x \in [a, b]\}. \end{aligned}$$

Remarks

1. These results require f to be continuous at *every* point of a *closed* interval. The results may fail if either of these conditions does not hold.

2. The last theorem asserts that not only is f bounded but that the lub and glb are ac-

tual values of f, i.e. f has a *maximum* and *minimum* on $[a, b]$.

3. The proofs we give make use of the least upper bound axiom. More illuminating proofs can be given in a metric space context.

Lemma (Sign-preserving property for continuous functions)
Let f be continuous at a and suppose $f(a) \neq 0$. Then there is an interval $(a - \delta, a + \delta)$ about a (i.e. $B(a, \delta)$) in which f has the same sign as $f(a)$.

Proof The argument is essentially the same as the one we used for limits of quotients. Suppose $f(a) > 0$. By continuity, for every $\epsilon > 0$ there exists $\delta > 0$ such that $f(a) - \epsilon < f(x) < f(a) + \epsilon$, whenever $a - \delta < x < a + \delta$. If we take the δ corresponding to $\epsilon = f(a)/2$ (a *positive* ϵ) then we get $\frac{1}{2}f(a) < f(x) < \frac{3}{2}f(a)$, whenever $a - \delta < x < a + \delta$. So $f(x) > 0$ in this interval and hence $f(x)$ and $f(a)$ have the same sign.

If $f(a) < 0$ we take the δ corresponding to $\epsilon = -f(a)/2$ to get the same result.

Proof of Bolzano's Theorem
Assume $f(a) < 0, f(b) > 0$. There may be many values of x in (a, b) for which $f(x) = 0$ but we shall find the *largest* one. Let $S = \{x \in [a, b] : f(x) \leq 0\}$. Clearly $S \neq \emptyset$ since $f(a) < 0$ (i.e. $a \in S$) and S is bounded above since $S \subset [a, b]$. So S has a least upper bound c. We shall prove that $f(c) = 0$.

There are only 3 possibilities: $f(c) > 0, f(c) < 0, f(c) = 0$. If $f(c) > 0$ there is an interval $(c - \delta, c + \delta)$, or $(c - \delta, c]$ if $c = b$, in which f is positive. So no point of S can lie to the right of $c - \delta$, i.e. $c - \delta$ is an upper bound for S which is *less* than the least upper bound. This contradiction shows that we can't have $f(c) > 0$.

If $f(c) < 0$ there is an interval $(c - \delta, c + \delta)$, or $[c, c + \delta)$ if $c = a$, in which f is negative. So $f(x) < 0$ for some $x > c$, which contradicts the fact that c is an upper bound for S. Hence $f(c) < 0$ is also impossible. So $f(c) = 0$ and $a < c < b$ since $f(a) < 0, f(b) > 0$.

Remark We deduce the Intermediate Value Theorem by applying the above result to the function $g(x) = f(x) - k$.

Proof of the Boundedness Theorem
We shall obtain a contradiction by a process of repeated bisection. Assume f is *unbounded* on $[a, b]$ and let c be the midpoint of $[a, b]$. Then f is unbounded on at least one of the subintervals $[a, c]$ and $[c, b]$. Let $[a_1, b_1]$ be the subinterval on which f is unbounded (or the left interval if there is a choice). Continue the bisection process deriving an interval $[a_{n+1}, b_{n+1}]$ from the interval $[a_n, b_n]$.

Note that we are constructing a family of nested closed intervals and the length of $[a_n, b_n]$ is $(b - a)/2^n$ which $\to 0$ as $n \to \infty$. These intervals intersect in a single point α by the Nested Intervals Theorem. Clearly $\alpha \in [a, b]$. By the continuity of f at α there is an interval $(\alpha - \delta, \alpha + \delta)$ in which $|f(x) - f(\alpha)| < 1$, i.e. $|f(x)| < 1 + |f(\alpha)|$. So f is bounded on this interval.

But $[a_n, b_n]$ lies inside $(\alpha - \delta, \alpha + \delta)$ when n is so large that $(b - a)/2^n < \delta$. So f is *bounded* on $[a_n, b_n]$. this contradicts the construction of $[a_n, b_n]$. so the original assumption that f is unbounded on $[a, b]$ is false.

Proof of the Extreme Value Theorem
By the previous theorem f is bounded on $[a, b]$, i.e. the range of f is a bounded set of numbers , and so has a least upper bound p and greatest lower bound q. We shall show that for some $c \in [a, b]$ we have $f(c) = p$.

Assume $f(x) \neq p$ for any $x \in [a, b]$ and let $g(x) = p - f(x)$. Then $g(x) > 0$ for all $x \in [a, b]$ and so $1/g$ is continuous on $[a, b]$. By the previous theorem we wee that $1/g$ is bounded on $[a, b]$. Say $1/g < C$ for all $x \in [a, b]$, where $C > 0$. Then $p - f(x) > 1/C$ and so $f(x) < p - 1/C$ for all $x \in [a, b]$. But this contradicts the definition of p as the lub of the range of f. So $f(x) = p$ for at least one $c \in [a, b]$.

The corresponding result about the glb follows by noting that $\text{glb}\{f(x) : x \in [a, b]\} = -\text{lub}\{-f(x) : x \in [a, b]\}$.