1. (a) Let $(G, *)$ be a group and H and K be subgroups of G. Prove that $H \cap K$ is a subgroup of G.

(b) Give two reasons why $H \cup K$ is not a subgroup of $(\mathbb{Z}_6, +_6)$, where $H = \{0, 2, 4\}$ and $K = \{0, 3\}$ are subgroups of $\mathbb{Z}_6$.

2. Define $*$ on $\mathbb{Z} \times \mathbb{Z}$ by $(a, b) * (c, d) = (a + c, b + d)$ for all $a, b, c, d \in \mathbb{Z}$. Prove that the map $\varphi : (\mathbb{Z} \times \mathbb{Z}, *) \to (\mathbb{Z}, +)$ defined by $\varphi((a, b)) = 3a - 6b$ is a homomorphism. What is the kernel of φ?

3. Let $(G, *)$ be an abelian group and $H = \{\, x \in G \mid x^3 = e \,\}$. Show that H is a subgroup of G.

4. (a) Prove that $G = \{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \mid a, b \in \mathbb{R}, a^2 + b^2 \neq 0 \,\}$ is a group together with matrix multiplication.

(b) Show that $(\mathbb{C} \setminus \{0\}, \cdot)$, the group of non-zero complex numbers together with multiplication, is isomorphic to $(G, \cdot)$.

5. The general linear group under matrix multiplication is defined by

$$GL(2, F) = \{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in F, ad - bc \neq 0 \,\},$$

where F is any field.

(a) Let $G = GL(2, \mathbb{R})$ and $H = \{\, A \in G \mid \det(A) = 7^j, j \in \mathbb{Z} \,\}$. Show that H is a subgroup of G.

(b) Find the inverse of $A = \begin{pmatrix} 2 & 4 \\ 2 & 3 \end{pmatrix}$ in $GL(2, \mathbb{Z}_7)$.