EXAMINATION FOR BA BSc ETC 2000

MATHEMATICS

Principles of Mathematics

(Time allowed: THREE hours)

NOTE: Answer ALL the questions. All questions carry equal marks.

1. For any integer n, let $A(n)$ be the statement

"If n is odd then $n^2 + n$ is even".

(a) Write down the contrapositive of $A(n)$.

(b) Write down the converse of $A(n)$.

(c) Write down the negation of $A(n)$.

(d) Is $A(n)$ true for some $n \in \mathbb{N}$? If so, give an example, if not give a proof.

(e) Is $A(n)$ true for every $n \in \mathbb{N}$? If so, give a proof, if not give a counterexample.

(f) Is the contrapositive of $A(n)$ true for some $n \in \mathbb{N}$? Is it true for all $n \in \mathbb{N}$? Give brief reasons for your answer.

(g) Is the converse of $A(n)$ true for some $n \in \mathbb{N}$? Is it true for all $n \in \mathbb{N}$? Give brief reasons for your answer.

2. Let $f : A \to B$ be a function. Recall that for $C \subseteq B$, $f^{-1}(C) = \{ x \in A : f(x) \in C \}$.

(a) Let $C, D \subseteq B$. Show that $f^{-1}(C \cup D) = f^{-1}(C) \cup f^{-1}(D)$.

(b) Let Λ be and arbitary indexing set. For each $\alpha \in \Lambda$ let $S_\alpha \subseteq B$. Show that

$$f^{-1}\left(\bigcap_{\alpha\in\Lambda} S_\alpha\right) = \bigcap_{\alpha\in\Lambda} f^{-1}(S_\alpha).$$

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3. Define the relation $\sqsubseteq$ on the Cartesian plane $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ by

$$(x, y) \sqsubseteq (u, v) \text{ if and only if } x \leq u \text{ and } y \leq v.$$

(a) Which of the following are true?

(i) $(1, 1) \sqsubseteq (2, 3)$.

(ii) $(1, -1) \sqsubseteq (2, -2)$.

(iii) $(1, 2) \sqsubseteq (2, 1)$.

(iv) $(2, 1) \sqsubseteq (1, 2)$.

(b) Prove that $\sqsubseteq$ is a partial order on $\mathbb{R}^2$.

(c) Is $\sqsubseteq$ a total order on $\mathbb{R}^2$? Give a brief reason for your answer.

4. A function $f : \mathbb{R} \to \mathbb{R}$ is said to be a *foo* function if $f(x + 1) = f(x) + 1$ for every $x \in \mathbb{R}$, and a *bar* function if $f(x + 1) = x - 1$ for all $x \in \mathbb{R}$.

(a) Show that if f and g are both foo functions then $f \circ g$ is a foo function.

(b) Prove that $f : \mathbb{R} \to \mathbb{R}$ is a foo function if and only if for every $x \in \mathbb{R}$ we have $f(x-1) = f(x)-1$.

(c) Show that if f is a foo function and g is a bar function then $f \circ g$ is a bar function.

5. (a) Prove that if n is a natural number then $1 + 3 + \cdots + (2n - 1) = n^2$.

(b) Prove that $2^{n+2} < 3^n$ for all integers $n \geq 4$.

6. Let $n \in \mathbb{N}$ with $n \geq 2$.

(a) Explain what it means to say that addition modulo n, $+_n$, is well-defined.

(b) Show that addition modulo n, $+_n$, is well-defined.

(c) Show that $(\mathbb{Z}_n, +_n)$ is a group.

7. Let $f : \mathbb{R} \to \mathbb{R}$ be an order-isomorphism, let $A \subseteq \mathbb{R}$, and let $x \in \mathbb{R}$. Show that x is a least upper bound for A if and only if $f(x)$ is a least upper bound for $f(A) = \{ f(y) : y \in A \}$.