1. (a) $(n^2 \leq 35) \Longrightarrow (n \leq 5)$

(b) $(n^2 > 35) \Longrightarrow (n > 5)$

(c) The negation of $A \Longrightarrow B$ is $A \wedge (\sim B)$ which gives $(n > 5) \wedge (n^2 \leq 35)$.

(d) Yes it's sometimes true.

Example 1: Let $n = 1$. Then the hypothesis is false and the statement is (vacuously) true.

Example 2: Let $n = 7$. Then $n^2 = 49 > 35$. Since the conclusion is true, then statement is true. (Note the hypothesis is also true in this case.)

(e) It is always true.

Proof: Let $n \in \mathbb{Z}$ satisfy $n > 5$. Then, since n is an integer, $n \geq 6$.

So $n^2 = n.n \geq 6.n \geq 6.6 = 36 > 35$, so $n^2 > 35$ as required.

(f) It is always true since it is equivalent to the original statement, proven in the previous part.

(g) The converse is sometimes true, but not always true.

Example 1: Let $n = 10$. Then the hypothesis $(n^2 > 35)$ is true and the conclusion $(n > 5)$ is also true.

Example 2: Let $n = -10$. Then the hypothesis $(n^2 > 35)$ is true but the conclusion $(n > 5)$ is false.

2. (a) Recall the relation is defined as follows. Let $(u, v), (x, y) \in \mathbb{R}^2$. Then $(u, v) \sim (x, y) \Longleftrightarrow u + y = x + v$. This is equivalent to $u - v = x - y$ which is an easier form to work with.

By definition, $T_{(0,0)} = \{(x, y) \in \mathbb{R}^2 : (x, y) \sim (0, 0)\}$.

So $(x, y) \in T_{(0,0)}$ iff $x - y = 0 - 0 = 0$, ie $x = y$.

Therefore $T_{(0,0)} = \{(x, x) : x \in \mathbb{R}\}$.

(b) By definition, $T_{(u,v)} = \{(x, y) \in \mathbb{R}^2 : (x, y) \sim (u, v)\}$.

So $(x, y) \in T_{(u,v)}$ iff $x - y = (u - v)$, ie $x = y + (u - v)$.

Therefore $T_{(u,v)} = \{(x, x - (u - v)) : x \in \mathbb{R}\}$ or (neater) $\{(y + u - v, y) : y \in \mathbb{R}\}$.

$T_{(u,v)}$ is a line in $\mathbb{R}^2$ with gradient 1 that crosses the y axis at $v-u$.

(c) $\mathcal{R}_\sim$ is, from the previous part, a set of lines in $\mathbb{R}^2$ with gradient 1.
By choosing say $\{T_{(0,v)} : v \in \mathbb{R}\}$ we get EVERY line with gradient 1.

(d) Basically there are 2 choices in this part; prove $\sim$ is symmetric, reflexive and transitive OR prove that $\mathcal{R}_\sim$ is a partition of $\mathbb{R}^2$ and use the theorem we proved in class.

Method 1. We prove that $\sim$ is reflexive, symmetric and transitive.

Reflexive: Let $(a,b) \in \mathbb{R}^2$. Clearly $a-b=a-b$, which implies that $(a,b) \sim (a,b)$ as required.

Symmetric: Let $(a,b),(c,d) \in \mathbb{R}^2$ satisfy $(a,b) \sim (c,d)$. This implies that $a-b=c-d$. So $c-d=a-b$, which implies that $(c,d) \sim (a,b)$ as required.

Transitive: Let $(a,b),(c,d),(e,f) \in \mathbb{R}^2$ satisfy $(a,b) \sim (c,d)$.and $(c,d) \sim (e,f)$.
The first relation implies that $a-b=c-d$, and the second implies that $c-d=e-f$.
So $a-b=e-f$, which implies that $(a,b) \sim (e,f)$ as required.

Since $\sim$ is symmetric, reflexive and transitive, it is by definition an equivalence relation.

Method 2. We need to show that $\mathcal{R}_\sim = \{T_{(u,v)} : (u,v) \in \mathbb{R}^2\}$ is a partition of $\mathbb{R}^2$. The definition of partition says that this is the case iff the following 2 properties are satisfied:

1. $\forall (p,q) \in \mathbb{R}^2, \exists T_{(u,v)} \in \mathcal{R}_\sim$ such that $(p,q) \in T_{(u,v)}$ and
2. $\forall (p,q),(u,v) \in \mathbb{R}^2, T_{(u,v)} \cap T_{(p,q)} \neq \phi \Longrightarrow T_{(u,v)} = T_{(p,q)}$.

Informally, 1 is true because every point in $\mathbb{R}^2$ has a line with gradient 1 passing through it, and 2 because if a pair of lines with the same gradient have a point in common, they are the same line! (I was willing to accept this argument.)

Proof of 1: Let $(p,q) \in \mathbb{R}^2$, and let $b=p-q$. Then the line $y=x+b$ (with gradient 1) contains the point (p,q). $\mathcal{R}_\sim$ is the set of all lines.with gradient 1.

Proof of 2: Let $(p,q),(u,v) \in \mathbb{R}^2$ and assume $(a,b) \in \mathbb{R}^2$ satisfies $(a,b) \in T_{(u,v)} \cap T_{(p,q)}$.
Then $(a,b) \sim (p,q)$.and $(c,d) \sim (u,v)$.
The first relation implies that $a-b=p-q$, and the second implies that $a-b=u-v$..
So $p-q=u-v$., which implies that as required.
$T_{(p,q)} = \{(x, x-(p-q)) : x \in \mathbb{R}\} = \{(x, x-(u-v)) : x \in \mathbb{R}\} = T_{(u,v)}$ as required.

3. (a) [First of all, notice that we **don't** assume that $(f \circ g)(y) = y$ for all $y \in B$, so we can't call g the inverse of f. Although you can use the ideas in the proof in your notes that if f has an inverse then f is one-to-one, you can't just copy that proof out indiscriminately.]

Suppose there is a function $g : B \to A$ such that $(g \circ f)(x) = x$ for all $x \in A$. We want to show that f is one-to-one, so let $x, y \in A$ with $f(x) = f(y)$ [we want to show that $x = y$]. Then $g(f(x)) = g(f(y))$, i.e. $(g \circ f)(x) = (g \circ f)(y)$, so $x = y$. Thus f is one-to-one.

(b) Suppose there is a function $h : B \to A$ such that $(f \circ h)(y) = y$ for all $y \in B$. We want to show that f is onto, so let $y \in B$ [we want to find $x \in A$ such that $f(x) = y$]. Now, $y = (f \circ h)(y) = f(h(y))$, so if we put $x = h(y)$ then $f(x) = y$, as required. Thus f is onto.

(c) [This is an "if and only if" proof, so we need to prove both implications.]

Suppose first that f has an inverse. Then, by part (a) we know that f is one-to-one and by part (b) we know that f is onto. So f is a one-to-one correspondence.

Conversely, suppose that f is a one-to-one correspondence. [We want to show that f has an inverse, in other words we are showing that the inverse exists, and to do this we give a definition of a function g and check that it is an inverse.]

Method 1: For any $y \in B$ we know that there is at least one $x \in A$ with $f(x) = y$, since f is onto, and that there is at most one such x, since f is one-to-one. So we can define $g : B \to A$ by declaring that $g(y)$ is the unique x in A satisfying $f(x) = y$. This gives us $f(g(y)) = y$. Also, for any x we have $g(f(x)) =$ the unique z satisfying $f(z) = f(x)$, namely $z = x$, so $g(f(x)) = x$. So g is an inverse of f.

Method 2: Since $f : A \to B$, we have $f \subseteq A \times B$. Let $g = \{ (b, a) \in B \times A : (a, b) \in f \}$. We must check that g is a function from B to A, and that g is an inverse of f.

g **is a function** if $(b, a_1) \in g$ and $(b, a_2) \in g$ then $f(a_1) = f(a_2) = b$, so since f is one-to-one we have $a_1 = a_2$.

$\mathrm{dom}(g) = B$ We have $\mathrm{dom}(g) \subseteq B$, and for every $b \in B$ there is an $a \in A$ with $(a, b) \in f$ (since f is onto), so there is an $a \in A$ with $(b, a) \in g$.

g **is an inverse of** f Let $a \in A$. Then $(a, f(a)) \in f$ so $(f(a), a) \in g$, so $g(f(a)) = a$. Similarly, $(b, g(b)) \in g$, so $(g(b), b) \in f$, so $f(g(b)) = b$. Thus g is an inverse of f.

4. (a) [We have to prove two implications: if x is maximal then $f(x)$ is maximal, and the converse.]

Suppose first that x is a maximal element of X [note that we don't know that X is totally ordered, so this does not imply that x is a greatest element of X]. We will show that $f(x)$ is a maximal element of Y. So suppose that $y \in Y$ with $f(x) \sqsubseteq y$ [we want $f(x) = y$]. Since f is onto, there is some z with $f(z) = y$. So now we have $f(x) \sqsubseteq f(z)$, so $x \leq z$. By maximality, we have $x = z$, so $f(x) = f(z)$, i.e. $f(x) = y$, as required. Thus $f(x)$ is a maximal element of Y.

Conversely, suppose that $f(x)$ is a maximal element of Y. Let $z \in X$ with $x \leq z$ [we want $x = z$]. Then $f(x) \sqsubseteq f(y)$, so $f(x) = f(z)$, so (since f is one-to-one) $x = z$, as required. Thus x is a maximal element of X.

(b) [Note: there are functions from $(0, 1)$ to $[0, 1]$ which are one-to-one, functions which are onto, and functions which are order-preserving. What we have to do is show that no function can be all three at once. We can use the result in part (a) to do this.]

Suppose $f : X \to Y$ were an order-isomorphism. Since $1 \in Y$, there is some $x \in X$ with $f(x) = 1$. Since 1 is maximal in Y, x must be maximal in X. But there is no maximal element in X. So there is no such isomorphism.

(c) [To show that two sets are isomorphic, we must give an isomorphism from one to the other. We define a function, and then check that it is indeed an order-isomorphism.]

Define $f : Z \to W$ by

$$f(x) = \begin{cases} x & \text{if } x \in (0,1], \\ x-1 & \text{if } x \in (2,3). \end{cases}$$

We will check that f is one-to-one, onto and order preserving.

f **is one-to-one:** Suppose $x, y \in Z$ with $f(x) = f(y)$ [we want to show that $x = y$]. BObviously, if $f(x) = x$ and $f(y) = y$ then this will imply that $x = y$, and likewise if $f(x) = x - 1$ and $f(y) = y - 1$. So the only thing that could go wrong is that we could have WLOG $x \in (0,1]$ and $y \in (2,3)$. But in that case we would have $f(x) \le 1$ and $f(y) = y - 1 > 2 - 1 = 1$, so we would not have $f(x) = f(y)$. SO the only way we can have $f(x) = f(y)$ is when $x = y$, as required.

f **is onto:** Let $y \in W$ [we want to find some $x \in Z$ with $f(x) = y$]. If $y \le 1$ then $y \in Z$ and $f(y) = y$. Otherwise, $y + 1 \in Z$ and $f(y + 1) = (y + 1) - 1 = y$. SO either way there is some $x \in Z$ with $f(x) = y$.

If $x \le y$ **then** $f(x) \le f(y)$**:** Suppose $x, y \in Z$ with $x \le y$. If x and y are both in $(0,1]$ then $f(x) = x$, $f(y) = y$, so $f(x) \le f(y)$. If $x \in (0,1]$, $y \in (2,3)$ then $f(x) = x \le 1$, $f(y) = y - 1 > 2 - 1 = 1$, so $f(x) \le f(y)$. And if x and y are both in $(2,3)$ then $x - 1 \le y - 1$, so $f(x) \le f(y)$. So in any case we have $f(x) \le f(y)$.

If $f(x) \le f(y)$ **then** $x \le y$**:** Suppose $x, y \in Z$ with $f(x) \le f(y)$. If x and y are both in $(0,1]$, then this obviously implies $x \le y$. If x and y are both in $(2,3)$ then this implies $x - 1 \le y - 1$, so $x \le y$. If $x \in (0,1]$ and $y \in (2,3)$ then we already know that $x \le y$. So the only thing that could go wrong is if $x \in (2,3)$, $y \in (0,1]$. But in that case we would have $f(y) < f(x)$. So that case can't occur: in other words, in any case which can occur we have $x \le y$, as required.