1. For $n \in \mathbb{N}$ let P_n be the statement "$8^n - 5^n$ is a multiple of 3". We prove by induction that P_n is true for all $n \in \mathbb{N}$.

Base: $(n = 1)$ We have $8^1 - 5^1 = 3 = 1 \cdot 3$, so P_1 is true.

Inductive step: Suppose $n \in \mathbb{N}$ and P_n is true. Then we can find some $k \in \mathbb{N}$ such that $8^n - 5^n = 3k$. So

$$\begin{aligned} 8^{n+1} - 5^{n+1} &= 8 \cdot 8^n - 5 \cdot 8^n + 5 \cdot 8^n - 5 \cdot 5^n \\ &= (8-5)8^n + 5(8^n - 5^n) \\ &= 3 \cdot 8^n + 5 \cdot 3k \\ &= 3(8^n + 5), \end{aligned}$$

so $8^{n+1} - 5^{n+1}$ is a multiple of 3, i.e. P_{n+1} is true.

Hence, by induction, P_n is true for all $n \in \mathbb{N}$. (4 marks)

2. (a) Let f be a wibble function. For each $n \in \mathbb{N}$ let P_n be the statement "for all $x \in \mathbb{R}$, $f(2^n x) = 2^n f(x)$". We prove by induction that P_n is true for all $n \in \mathbb{N}$.

Base: P_1 is the assertion that for all $x \in \mathbb{R}$, $f(2x) = 2f(x)$, which is the defintion of a wibble function.

Inductive step: Suppose $n \in \mathbb{N}$ and P_n is true. We must show that P_{n+1} is true. So let $x \in \mathbb{R}$. Then

$$\begin{aligned} f(2^{n+1}x) &= f(2 \cdot 2^n x) \\ &= 2f(2^n x) && \text{since } f \text{ is a wibble function} \\ &= 2 \cdot 2^n f(x) && \text{by inductive hypothesis} \\ &= 2^{n+1} f(x), \end{aligned}$$

so P_{n+1} is also true, as required.

Hence, by induction, P_n is true for all $n \in \mathbb{N}$, as required. (4 marks)

(b) Suppose that f and g are both wibble functions. Let $x \in \mathbb{R}$. Then

$$(f+g)(2x) = f(2x) + g(2x) = 2f(x) + 2g(x) = 2(f+g)(x).$$

Thus $f + g$ is a wibble function. (2 marks)

(c) The converse of the result in part (b) is "If $f + g$ is a wibble function then f and g are both wibble functions." (1 mark)

(d) To show that the converse of the result in part (b) is false, we give an example of a non-wibble function f and a function g such that $f + g$ is a wibble function.[1] Define $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} x & \text{if } x \in [0,1] \\ 0 & \text{if } x \in \mathbb{R} \setminus [0,1] \end{cases} \qquad g(x) = \begin{cases} 0 & \text{if } x \in [0,1] \\ x & \text{if } x \in \mathbb{R} \setminus [0,1] \end{cases}$$

[1] Note: it is also the case that g is not a wibble function, but we don't need to prove that in order to answer the question

Now $f(1) = 1$ but $f(2) = 0 \neq 2f(1)$. Thus f is not a wibble function. However, $(f + g)(x) = x$, and it is easy to see that $h(x) = x$ is a wibble function: $h(2x) = 2x = 2h(x)$. (4 marks)

3. For $b \in \mathbb{N}$ let P_b be the statement "For any $a \in \mathbb{N}$, if $d = \gcd(a, b)$ then there exist integers x and y with $d = ax + by$." We prove by complete induction that P_b is true for all $b \in \mathbb{N}$.

Base: Let $b = 1$, and let $a \in \mathbb{N}$. Let $d = \gcd(a, b)$. Then $d = a0 + bd$.

Inductive step: Suppose $b \in \mathbb{N}$ and $P_1, P_2, \ldots, P_b$ are all true. We must show that P_{b+1} is true. So let $a \in \mathbb{N}$, and let $d = \gcd(a, b + 1)$. By the Division Algorithm, we can find $q, r \in \mathbb{Z}$ with $a = q(b + 1) + r$ and $0 \leq r < b + 1$. Since $r < b + 1$, $r \leq b$. There are two cases to consider, depending on whether $r = 0$ or not. If $r = 0$ then $b + 1 \mid a$, so $d = b+1 = a0 + (b+1)1$. Otherwise, we have $1 \leq r \leq b$, so by the inductive hypothesis P_r is true. Also, $d = \gcd(a, b + 1) = \gcd(b + 1, r)$, so by P_r we know that there exist $x, y \in \mathbb{Z}$ with $d = (b + 1)x + ry$. Substituting $r = a - q(b + 1)$ we get

$$d = (b + 1)x + (a - q(b + 1))y = ay + (b + 1)(x - qy),$$

and y and $x - qy$ are both integers. Thus P_{b+1} is also true.

Hence, by complete induction, P_b is true for all $b \in \mathbb{N}$. (6 marks)

4. Let $a, b \in \mathbb{N}$. Suppose first that a and b are relatively prime. Then $\gcd(a, b) = 1$, so by Question 3 there exist integers x and y such that $ax + by = 1$. (1 mark)

Conversely, suppose that there exist integers x and y such that $ax+by = 1$. We must show that a and b are relatively prime, in other words we must show that if d is a positive integer with $d \mid a$ and $d \mid b$ then $d = 1$. So suppose d is a positive integer with $d \mid a$ and $d \mid b$. Then there exist integers p and q with $a = pd$ and $b = qd$. Substituting this into $ax+by = 1$ gives $pdx+qdy = 1$, so $(px + qy)d = 1$. Since $px + qy$ and d are integers, this implies that $px + qy = d = 1$ or $px + qy = d = -1$. Since d is positive, we must have $d = 1$, as required. (4 marks)

5. We apply Euclid's Algorithm to 53 and 25:

$$\begin{aligned} 53 &= 2 \cdot 25 + 3 \\ 25 &= 8 \cdot 3 + 1 \\ 8 &= 8 \cdot 1 + 0 \end{aligned}$$

Since 1 is the last non-zero remainder, we have $\gcd(53, 25) = 1$, i.e. 53 and 25 are relatively prime. (2 marks)

From the first two lines of Euclid's Algorithm, we get $3 = 53-2\cdot25$ and $1 = 25-8\cdot3$. Substituting the first into the second, we get $1 = 25 - 8(53 - 2 \cdot 25)$, so $1 = 53(-8) + 25 \cdot 17$. (3 marks)