1. (a) Suppose (x_n) and (y_n) are both increasing. Let $n, m \in \mathbb{N}$ with $n \leq m$. Then $x_n \leq x_m$ and $y_n \leq y_m$, so $x_n + y_n \leq x_m + y_m$, i.e. $z_n \leq z_m$. Thus (z_n) is an increasing sequence. (1 mark)

The converse of the statement is "If (z_n) is an increasing sequence then (x_n) and (y_n) are both increasing sequences". The converse does not hold. For a counterexample, we could have $x_n = n$, $y_n = \frac{1}{n}$ for each $n \in \mathbb{N}$. Then $z_n = n + \frac{1}{n}$, so (z_n) is an increasing sequence, but (y_n) is not an increasing sequence. (1 mark)

(b) We have seen that if (x_n) and (y_n) are both increasing then (z_n) is increasing, and a similar argument shows that if both sequences are decreasing then (z_n) will be decreasing. So we will need to have one sequence increasing and the other decreasing. One possible example would be $x_n = 4n$ $y_n = -n^2$. These give us $z_1 = 4 - 1 = 3$, $z_2 = 8 - 4 = 4$, $z_3 = 12 - 9 = 3$, $z_4 = 16 - 16 = 0, \ldots$. Thus the sequence z_n increases initially and then decreases, so it is not monotonic. (2 marks)

2. (a) Let $x \in A \triangle (B \triangle C)$. [We want to show that $x \in (A \triangle B) \triangle C$] Then $x \in A \setminus (B \triangle C)$ or $x \in (B \triangle C) \setminus A$

Case 1: $x \in A \setminus (B \triangle C)$. We have $x \notin B \triangle C$, so either $x \notin B \cup C$ or $x \in B \cap C$.

Case 1a: $x \in A \setminus (B \triangle C)$ and $x \notin B \cup C$. Then $x \in A \setminus B$, so $x \in A \triangle B$, and $x \notin C$. Thus $x \in (A \triangle B) \triangle C$ in this case.

Case 1b: $x \in A \setminus (B \triangle C)$ and $x \in B \cap C$. Then $x \in A \cap B$, so $x \notin A \triangle B$, and $x \in C$. Thus $x \in C \setminus (A \triangle B)$, so $x \in (A \triangle B) \triangle C$ in this case also.

Case 2: $x \in (B \triangle C) \setminus A$. Then $x \in (B \setminus C) \cup (C \setminus B)$.

Case 2a: $x \in B \setminus C$. Since we also have $x \notin A$, we have $x \in A \triangle B$ and $x \notin C$, so $x \in (A \triangle B) \triangle C$ in this case.

Case 2b: $x \in C \setminus B$. Then $x \notin A \cup B$ so $x \notin A \triangle B$, and $x \in C$, so $x \in (A \triangle B) \triangle C$ in this case.

So, in any case we have $x \in (A \triangle B) \triangle C$. Thus $A \triangle (B \triangle C) \subseteq (A \triangle B) \triangle C$. (3 marks)

Now we have to show the converse. Let $y \in (A \triangle B) \triangle C$: we will show that $y \in A \triangle (B \triangle C)$. Again we have four cases to consider.

Case 1: $y \in (A \triangle B) \setminus C$.

Case 1a: $y \in A \setminus B$ and $y \notin C$. Then $y \in A \setminus (B \triangle C)$ so $y \in A \triangle (B \triangle C)$.

Case 1b $y \in B \setminus A$ and $y \notin C$. Then $y \in (B \setminus C$ so $y \in B \triangle C$, and $y \notin A$ so $y \in A \triangle (B \triangle C)$.

Case 2: $y \in C \setminus (A \triangle B)$.

Case 2a: $y \notin A \cup B$. Then $y \notin (A \triangle B)$ and $y \in C$, so $y \in A \triangle (B \triangle C)$.

Case 2b: $y \in A \cap B$. Then $y \in A$ and $y \in B \cap C$ so $y \notin B \triangle C$. Thus $y \in A \triangle (B \triangle C)$.

So, in any case, we have $y \in A \triangle (B \triangle C)$. Thus $(A \triangle B) \triangle C) \subseteq A \triangle (B \triangle C)$, as required. (2 marks)

(b) We must check five things: that $\triangle$ is an operation on $\mathcal{P}(S)$; that $\triangle$ is associative; that there is an identity element; that every element has an inverse; and that $\triangle$ is commutative.

First, note that if $A, B \subseteq S$ then $A \setminus B \cup B \setminus A \subseteq A \cup B \subseteq S$, so if $A, B \in \mathcal{P}(S)$ then $A \triangle B \in \mathcal{P}(S)$. So $\triangle$ really is a binary operation on $\mathcal{P}(S)$.

Part (a) shows that $\triangle$ is associative.

We have $A \triangle \varnothing = (A \setminus \varnothing) \cup (\varnothing \setminus A) = A$ and $\varnothing \triangle A = (\varnothing \setminus A) \cup (A \setminus \varnothing) = A$ for all A, so $\varnothing$ is an identity element.

We have $A \triangle A = (A \setminus A) \cup (A \setminus A) = \varnothing$, so every element has an inverse (namely $A^{-1} = A$ for each A).

Finally, note that if $A, B \in \mathcal{P}(S)$ then $A \triangle B = (A \smallsetminus B) \cup (B \smallsetminus A) = (B \smallsetminus A) \cup (A \smallsetminus B) = B \triangle A$, so $\triangle$ is a commutative operation.

So $(\mathcal{P}(S), \triangle)$ is an abelian group, as required. (3 marks)

3. Suppose first that H is a subgroup of G. Then H has an identity element, so there is some $e' \in H$ which satisfies $e' * h = h$ for all $h \in H$. In particular we have $e' * e' = e'$. But we know that the only solution of $x * x = x$ in G is $x = e$, so we have $e' = e$. Thus $e \in H$. Now, if $x, y \in H$ then, since $*$ is an operation on H we must have $x * y \in H$. Finally, if $x \in H$ then x has an inverse in H, so there is some $y \in H$ with $x * y = y * x = e$. The only such $y \in G$ is $y = x^{-1}$, so we have $x^{-1} \in H$. (2 marks)

Conversely, suppose that $H \subseteq G$ with $e \in H$, $x * y \in H$ for every $x, y \in H$, and $x^{-1} \in H$ for every $x \in H$. We must show that H is a subgroup of G, i.e. we must show that $*$ is a group operation on H. By the second assumption we know that $*$ is an operation on H. Since $x * (y * z) = (x * y) * z$ for all $x, y, z \in G$, we certainly know that the same holds for all $x, y, z \in H$, so $*$ is an associative operation on H. Similarly, we know that $x * e = e * x = x$ for all $x \in G$, and in particular for all $x \in H$, so e is an identity element for H. Finally, for any $x \in H$ we have $x^{-1} \in H$ and $x * x^{-1} = x^{-1} * x = e$, so every element has an inverse in H. Thus $*$ is indeed a group operation on H, so H is a subgroup of G. (4 marks)

4. Suppose first that H is a subgroup of G. By Question 3 we know that $e \in H$, so $H \neq \varnothing$. Let $x, y \in H$: we must show that $x * y^{-1} \in H$. Now, by Question 3 we know that $y^{-1} \in H$, so by Question 3 again we know that $x * y^{-1} \in H$. (2 marks)

Conversely, suppose that $H \neq \varnothing$ and, for every $x, y \in H$ we have $x * y^{-1} \in H$. We will show that H is a subgroup. By Question 3 again, it is enough to show that $e \in H$, $x^{-1} \in H$ for every $x \in H$, and $x * y \in H$ for every $x, y \in H$. First, note that since $H \neq \varnothing$, there is some $z \in H$. By hypothesis, we have $z * z^{-1} \in H$, i.e. $e \in H$. Now let $x \in H$. Then, since we also know that $e \in H$ we have $e * x^{-1} = x^{-1} \in H$. Finally, let $x, y \in H$. Then by the previous line we know that $y^{-1} \in H$, so by hypothesis we have $x * (y^{-1})^{-1} \in H$. But $(y^{-1})^{-1} = y$, so $x * y \in H$, as required. (5 marks)

5. We can use either the characterisation in Question 3 or that in Question 4. The first is probably easier in this case. So we will check that $e_G \in \ker(f)$, that if $x, y \in \ker(f)$ then $x * y \in \ker(f)$, and that if $x \in \ker(f)$ then $x^{-1} \in \ker(f)$.

Note that $e_G * e_G = e_G$, so $f(E_G) \diamond f(e_G) = f(e_G)$. The only solution of $h \diamond h = h$ is $h = e_H$, so we have $f(e_G) = e_H$, and therefore $e_G \in \ker(f)$.

Now let $x, y \in \ker(f)$. Then $f(x * y) = f(x) \diamond f(y) = e_H \diamond e_H = e_H$, so $x * y \in \ker(f)$.

Finally, let $x \in \ker(f)$. Then

$$f(x^{-1}) = e_H \diamond f(x^{-1}) = f(x) \diamond f(x^{-1}) = f(x * x^{-1}) = f(e_G) = e_H,$$

so $x^{-1} \in \ker(f)$, as required.

Thus, by Question 3, $\ker(f)$ is a subgroup of G. (5 marks)