**1.** (a) Maximal Elements: $\{1, 4\}$ Let $b \in \{1, 4\}$. From the lattice diagram (or checking the relation) there is no $x \in X$ other than $b$ that satisfies $b \leq x$.

Minimal Elements: $\{2\}$. From the lattice diagram (or checking the relation) there is no $x \in X$, $x \neq 2$ such that $x \leq 2$..

Greatest Elements: $\phi$ (none). Clearly the greatest element must be 1 or 4. But neither $1 \leq 4$ nor $4 \leq 1$ is true, so neither is a greatest element.

Least Element: $\{2\}$. $\forall x \in X$, $2 \leq x$.

(b) $A$ is a maximal element.

Proof: Clearly $A \in \mathcal{P}(A)$. Assume that there is a $B \in \mathcal{P}(A)$ satisfying $A \subseteq B$. Since $B \in \mathcal{P}(A)$, we also have $B \subseteq A$. This means that $x \in A \iff x \in B$. So $A = B$. So the only element of $\mathcal{P}(A)$ satisfying $A \subseteq B$ is $A$ as required.

$\mathcal{P}(A)$ has greatest element $A$ since (by definition) $\forall B \in \mathcal{P}(A), B \subseteq A$.

Total ordering example: $A = \{1\}$. Then $\mathcal{P}(A) = \{\phi, \{1\}\}$, and $\phi \subseteq \{1\}$. (The other conditions such as reflexivity are already assumed, since we've been given that $\subseteq$ is a partial ordering.)

Not a total ordering example: $A = \{1, 2\}$. $\{1\}, \{2\} \in \mathcal{P}(A)$ but they cannot be compared. In other words, neither $\{1\} \subseteq \{2\}$ nor $\{1\} \subseteq \{2\}$.

**2.** $T_1 = \{b \in \mathbb{Z} : (1-b)/3 \in \mathbb{Z}\}$ by the definition of $\sim$. So assume $(1-b)/3 = -k \in \mathbb{Z}$. Then $b = 3k+1$ where $k \in \mathbb{Z}$. So $T_1 = \{3k + 1 : k \in \mathbb{Z}\}$.

Similar calculations show that $T_0 = \{3k : k \in \mathbb{Z}\}$ and $T_2 = \{3k + 2 : k \in \mathbb{Z}\}$. Since every integer can be written as $3k$, $3k + 1$ or $3k + 2$, $\mathbb{Z} = T_0 \cup T_1 \cup T_2$.

We show next that these 3 sets are the *only* sets in $\mathcal{R}_\sim$. The easiest way is to first prove that $\sim$ is symmetric. Assume $a \sim b$. Then $(b - a)/3 = -(a - b)/3 \in \mathbb{Z}$, so $b \sim a$.

This implies $a \in T_b \iff b \in T_a$. So, since $\mathbb{Z} = T_0 \cup T_1 \cup T_2$, $\mathcal{R}_\sim = \{T_0, T_1, T_2\}$.

To show that $\sim$ is an equivalence relation, we need to show that $\mathcal{R}_\sim$ is a partition of $\mathbb{Z}$. We have already shown that the union of the sets in $\mathcal{R}_\sim$ is all of $\mathbb{Z}$. They are also disjoint since every integer can be written in *exactly* one of the ways given. Thus $\mathcal{R}_\sim$ is a partition of $\mathbb{Z}$ and $\sim$ is an equivalence relation.

The equivalence classes are $T_0, T_1$ and $T_2$. (Alternatively, they are the integers congruent to 0, 1 and 2 mod 3.)

**3.** We use the alternative definition of 1-1 (5.1.10 in the book) and prove its contrapositive:

$\forall x, y \in A$, $x \neq y \implies (g \circ f)(x) \neq (g \circ f)(y)$

Assume $x, y \in A$, $x \neq y$. Since $f$ is 1-1, $f(x) \neq f(y)$. Then, since $g$ is 1-1, $(g \circ f)(x) \neq (g \circ f)(y)$ as required.

Proof of $f(X) \backslash f(Y) \subseteq f(X \backslash Y)$:

Assume $b \in f(X) \backslash f(Y)$. Then $\exists x \in X$ such that $f(x) = b$, and there is no $y \in Y$ such that $f(y) = b$. So, in particular, $x \notin Y$. Therefore $x \in X \backslash Y$. So $b = f(x) \in f(X \backslash Y)$ as required.

One way to find an example is to cook up a function $f : A \to B$ whose domain contains 2 points $x$ and $y$ with the following properties.

1. $\exists b \in B$ such that $f^{-1}(\{b\}) = \{x, y\}$ (Inverse image of a set, NOT an inverse function !!)
2. $x \in X \backslash Y$ and $y \in Y \backslash X$.

The simplest example is $X = \{x\}$, $Y = \{y\}$, $A = \{x, y\}$, $B = \{b\}$, $f = \{(x, b), (y, b)\}$.

$X \backslash Y = X$, so $f(X \backslash Y) = \{b\}$. But $f(X) = f(Y) = \{b\}$, so $f(X) \backslash f(Y) = \phi$.

**4.** $f : [0, 1) \to [1, 3)$, $f(x) = 2x + 1$.

To prove that $f$ is a 1-1 correspondence, we need to show that it is both 1-1 and onto.

Let $x, y \in [1, 3)$ satisfy $f(x) = f(y)$. Then $2x + 1 = 2y + 1$. This implies that $x = y$, so $f$ is 1-1.

Let $z \in [1, 3)$. To prove that $f$ is onto, we need to prove that $\exists x \in [0, 1)$ such that $f(x) = z$. It is easy enough to 'solve 'for $x$, to obtain $x = (z - 1)/2$.

Finally, we need to check that $x \in [0, 1)$. Since $z \in [1, 3)$, $1 \leq z < 3$, so $0 \leq z - 1 < 2$, and $0 \leq (z - 1)/2 < 1$. So $x = (z - 1)/2 \in [0, 1)$ as required.

To prove that $f$ is an order isomorphism, we need to show that $f$ is 1-1 and onto (done already) and that $\forall x, y \in [0, 1)$, $x \leq y \iff f(x) \leq f(y)$.

$\implies$ Assume $x \leq y$. Then $2x \leq 2y$, and $f(x) = 2x + 1 \leq 2y + 1 = f(y)$ as required.

$\impliedby$ Assume $f(x) \leq f(y)$. Then $2x + 1 \leq 2y + 1$, and $2x \leq 2y$, so $x \leq y$ as required.

(NOTE: The two parts of this proof are really the same calculation 'run' in opposite directions. So we could combine the two into a single argument using 'is equivalent to ' or 'if and only if ' rather than 'implies '. In general, this is a dangerous practice; the two directions usually require different arguments. For example if the rule for $f$ was $f(x) = 2x^2 + 1$ you would need to be *very* careful to justify yourself.)

**5.** The set difference binary operation is neither associative nor commutative.

Let $A = \{1, 2\}$, $B = C = \{2\}$.

Associative: $(A\backslash B)\backslash C = \{1\}\backslash\{2\} = \{1\}$. However, $A\backslash(B\backslash C) = A\backslash\phi = \{1, 2\} \neq (A\backslash B)\backslash C$. So \ is not associative.

Commutative: $A\backslash B = \{1\}$. But $B\backslash A = \phi \neq A\backslash B$. So \ is not commutative.