NB: Please deposit your solutions in the appropriate box **by 4 p.m. on the due date.** Late assignments or assignments placed into incorrect boxes will not be marked. Use a mathematics department cover sheet. These are available from outside the Resource Centre. PLEASE SHOW ALL WORKING.

1. $\mathcal{P}(A) = \{\phi, \{0\}, \{1\}, \{0,1\}\}$.

$\mathcal{P}(B) = \{\phi, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$.

$\mathcal{P}(A) \cap \mathcal{P}(B) = \{\phi, \{1\}\} = \mathcal{P}(A \cap B)$ since $A \cap B = \{1\}$.

$\mathcal{P}(A) \cup \mathcal{P}(B) \neq \mathcal{P}(A \cup B)$ because $\{0,3\} \in \mathcal{P}(A \cup B)$ but $\{0,3\} \nsubseteq A$ and $\{0,3\} \nsubseteq B$, so $\{0,3\} \notin \mathcal{P}(A) \cup \mathcal{P}(B)$.

(1) $\mathcal{P}(X) \subseteq \mathcal{P}(Y) \Longrightarrow X \subseteq Y$.

Assume $\mathcal{P}(X) \subseteq \mathcal{P}(Y)$. Then every subset of X (including X itself) is a subset of Y.

(2) $\mathcal{P}(X) \neq \mathcal{P}(Y) \Longrightarrow X \neq Y$.

The contrapositive is $X = Y \Longrightarrow \mathcal{P}(X) = \mathcal{P}(Y)$ which is obviously true.

$\mathcal{P}(X) \subset \mathcal{P}(Y) \Longrightarrow X \subset Y$.

Assume $\mathcal{P}(X)$ is a proper subset of $\mathcal{P}(Y)$. Then (by definition) $\mathcal{P}(X) \subseteq \mathcal{P}(Y)$ and $\mathcal{P}(X) \neq \mathcal{P}(Y)$. (1) implies that $X \subseteq Y$ and (2) implies that $X \neq Y$. Thus (by the definition again) $X \subset Y$ as required.

2. (a) $C \subseteq B \Longrightarrow A \times C \subseteq A \times B$

Assume $C \subseteq B$. We need to show that $p \in A \times C \Longrightarrow p \in A \times B$. By the definion of Cartesian product, $p \in A \times C$ is an ordered pair (x, y) with $x \in A$ and $y \in C$. But $y \in B$ since $C \subseteq B$, so (by the definion of Cartesian product again), $p \in A \times B$ as required.

Picture: see attached page.

(b) Picture: see attached page.

The real number a is related to the real number b if and only if $a-b$ is an even natural number.

3.

Relation	Reflexive	Symmetric	Antisymmetric	Transitive
a	F	F★	T	T
b	F	F	T★	T
c	F	T	F	F
d	F	T	F	F★
e	T★	F	T	T

Relation (a) is not symmetric.

Counterexample: Let $a = 1$ and $b = 2$. Then $a \nsim b$ since $a - b = -1 \notin \mathbb{N}$. However $a \sim b$ since $b - a = 1 \in \mathbb{N}$.

Relation (b) is antisymmetric.

Proof: Assume (for a contradiction) that $a \sim b$ and $b \sim a$. Then each is a proper subset of the other; in particular, each is a subset of the other. This implies (using the defition of subset) that $x \in a \iff x \in b$. In other words, $a = b$. But $a \sim b$ implies that a is a *proper* subset of b, so $a \neq b$. This contradiction implies that NO sets satisfy the hypothesis in the definition of antisymmetric. (Look it up!) So the definition is (vacuously) satisfied.

Relation (d) is not transitive.

Counterexample: Let $a = c = 1$, and let $b = -1$. Then $a \sim b$ (since $1 + -1 = 0$) and $b \sim c$. However $a \nsim c$ (since $1 + 1 \neq 0$), so the relation is not transitive.

Relation (e) is reflexive.

Proof: Let $a \in \mathbb{Z}$. Then $a - a = 0 < 1$, so $a \sim a$ as required.

Note: The last row of the table is a bit of a trick question. We first rewrite the relation's definition, by noting that (because it is a relation on the INTEGERS) $a - b < 1 \iff a - b \leq 0 \Leftrightarrow a \leq b$. So the relation is just the usual total ordering on the integers. This would not be true if we used the same rule to define a relation on $\mathbb{R}$. The set a relation applies to is an important part of the relation's definition. Brownie points if you saw this!

4. The pairs given in S yield the first lattice diagram on the attached page. To give a reflexive relation, we must add the pairs $(1,1), (2,2), (3,3), (4,4)$ and $(5,5)$. To be transitive, the relation must contain $(4,2)$. (Look at the picture.)

So the minimal relation is $T = \{(1,1), (1,2), (2,2), (3,3), (4,1), (4,2), (4,3), (4,4), (5,2), (5,5), \}$. Note: there should be exactly this 10 pairs.

One total order is given by adding the pairs $(4,5), (1,5), (1,3), (5,3)$ and $(3,2)$. (Thecorresponding ordering is- listing the elements in order- 41532. Check that it is compatible with the previous lattice diagram.)

This gives

$U = \{(1,1), (1,2), (1,3), (1,5), (2,2), (3,2), (3,3), (4,1), (4,2), (4,3), (4,4), (4,5), (5,2), (5,3), (5,5)\}$.

Note that this relation has 15 ordered pairs; 5 to make it reflexive plus $(5 \times 4)/2$ to ensure that all pairs of distinct elements can be compared.