1. Table of answers to the 'little 'questions

	S'ment	Pred	Free Var	Bound Var	Thm	Taut	Cont'n
a	y	n	none	none	n	n	n
b	y	n	none	none	n	n	n
c	y	n	none	x,y,t	n(?)	n	n(?)
d	y	n	none	x,y,t	n(?)	n	n(?)
e	y	n	none	a,b,c	y	n	n

Note: in parts c and d I'll accept any plausible argument based on the properties of 'people' and the 'loves' 'relationship as long as they demonstrate an understanding of the logical structure of the statement!

(a) $(A \vee B) \wedge \sim (A \wedge B)$ or $(A \wedge \sim B) \vee (\sim A \wedge B)$ or $(A \vee B) \wedge (\sim A \vee \sim B)$. Other forms are possible,check the truth table. (Fewer marks for unnecessarily complex answers.)

A	B	Statement
T	T	F
T	F	T
F	T	T
F	F	F

(Table not required in the solution, but recommended.)

(b) $(A \wedge \sim B \wedge \sim C) \vee (\sim A \wedge B \wedge \sim C) \vee (\sim A \wedge \sim B \wedge C) \vee (A \wedge B \wedge C)$. A bonus mark for seeing how to simplify this John McK can't. Again, check the truth table.

A	B	C	Statement
T	T	T	T
T	T	F	F
T	F	T	F
T	F	F	T
F	T	T	F
F	T	F	T
F	F	T	T
F	F	F	F

(Table not required in the solution, but recommended.)

(c) $\forall x \in P, \exists y \in P, \exists t \in T$ such that $R(x, y, t)$.
(The commas and 'such that' are recommended but not necessary. All the symbols are necessary.)

(d) $\sim(\exists x \in P, \forall y \in P, \forall t \in T$ such that $R(x,y,t))$ or (better) $\forall x \in P, \exists y \in P, \exists t \in T$ such that $\sim R(x,y,t)$. (Comments as for part c.)

(e) $\forall a \in \mathbb{Z}, \forall b \in \mathbb{Z}, R(a,b) \implies (\exists c \in \mathbb{Q}$ such that $R(a,c) \wedge R(c,b))$.

(Comments as for previous 2 parts, but marks lost for not using the symbols $\implies$, $R(,)$ and $\wedge$.)

Proof: Let $a, b \in \mathbb{Z}$ satisfy $R(a,b)$, in other words $a < b$. Define $c = (a+b)/2$. Clearly $c \in \mathbb{Q}$. We claim that $R(a,c) \wedge R(c,b)$ is true; in other words $a < c$ and $c < b$. $c = (a+b)/2 < (b+b)/2 = b$ since $a < b$, and $a = (a+a)/2 < (a+b)/2 = c$ since $a < b$. So the claim is established, since c satisfies the requirements of the theorem.

2. $(A \implies B) \wedge \sim(A \iff C)$

A	B	C	$A \implies B$	$\sim(A \iff C)$	Statement
T	T	T	T	F	F
T	T	F	T	T	T
T	F	T	F	F	F
T	F	F	F	T	F
F	T	T	T	T	T
F	T	F	T	F	F
F	F	T	T	T	T
F	F	F	T	F	F

$(A \implies B) \wedge \sim(A \iff C) \quad \iff \quad (\sim A \vee B) \wedge \sim((\sim A \vee C) \wedge (A \vee \sim C))$

$\iff \quad (\sim A \vee B) \wedge (\sim(\sim A \vee C) \vee \sim(A \vee \sim C)) \iff \quad (\sim A \vee B) \wedge ((A \wedge \sim C) \vee (\sim A \wedge C))$.

Hmmm. Can we go further?? One approach is to prove "distributive" laws for $\vee$ and $\wedge$ and keep expanding.

Markers note; only give credit for this if the rules are proven or a source referenced! Not covered in text or in class. Another way is to read off the expression from the truth table. (One bracket for every line that ends in a T. Think about it.)

$$(A \wedge B \wedge \sim C) \vee (\sim A \wedge B \wedge C) \vee (\sim A \wedge \sim B \wedge C)$$

$\iff \quad (A \wedge B \wedge \sim C) \vee (\sim A \wedge C)$ (since $(X \wedge Y) \vee (\sim X \wedge Y) \iff Y$ This needs to be justified using a truth table!)

$\sim(\sim(A \vee \sim B) \wedge \sim(\sim A \vee B))$ is a tautology, and can be expressed as (for example) $A \vee \sim A$.

A	B	$\sim(A \vee \sim B)$	$\sim(\sim A \vee B)$	Statement
T	T	F	F	T
T	F	F	T	T
F	T	T	F	T
F	F	F	F	T

3. (a) $\Rightarrow$

Let $n \in \mathbb{Z}$ be divisible by 3. Then $n = 3k$ for some integer k, and $n^3 = 27k^3 = 3(9k^3)$. Therefore, since $9k^3 \in \mathbb{Z}$, n^3 is divisible by 3 as required.

$\Longleftarrow$

It is easiest to prove the contrapositive of the statement, which is "3 does not divide $n \Longrightarrow 3$ does not divide n^3"

There are 2 cases to check; $n = 3k + 1$ and $n = 3k + 2$.

Case1: $(3k + 1)^3 = 27k^3 + 27k^2 + 9k + 1 = 3(9k^3 + 9k^2 + 3k) + 1$ which is not divisible by 3 since $9k^3 + 9k^2 + 3k \in \mathbb{Z}$.

Case2: $(3k + 2)^3 = 27k^3 + 54k^2 + 36k + 8 = 3(9k^3 + 18k^2 + 12k + 2) + 2$ which is not divisible by 3 since $9k^3 + 18k^2 + 12k + 2 \in \mathbb{Z}$.

(Note that we use the "in exactly one way" part of the facts I said you could use here.)

(b) Assume, for a contradiction, that there is a rational number d that satisfies $d^3 = 3$. Then we may assume that $d = p/q$ where $p, q \in \mathbb{Z}$ have no common factor.

Then $3 = d^3 = p^3/q^3$, so

$p^3 = 3q^3$. (*)

This implies (since $q^3 \in \mathbb{Z}$) that p^3 is divisible by 3. From part a of this question, we conclude that p is divisible by 3. Let $p = 3t$, where $t \in \mathbb{Z}$.

Substituting in (*) we obtain $3q^3 = 27t^3$ or $q^3 = 3(3t^3)$. This implies that q^3 is divisible by 3 and (using part a again) that q is divisible by 3. Thus BOTH p and q are divisible by 3; in other words they have a common factor. This gives a contradiction to our original assumptions, and proves the result.

4. (a) Prove $A\backslash(A\backslash B) = A \cap B$. We prove $x \in A\backslash(A\backslash B) \Longleftrightarrow x \in A \cap B$.

$\Longrightarrow$ Let $x \in A\backslash(A\backslash B)$. Then $x \in A$ and $x \notin A\backslash B$. But $A\backslash B = \{x \in A : x \notin B\}$. Therefore we must have $x \in B$. Since x is in both A and B, $x \in A \cap B$ as required.

$\Longleftarrow$ Let $x \in A \cap B$. Then $x \in A$ and $x \in B$. So $x \notin A\backslash B$. Therefore, since $x \in A$, $x \in A\backslash(A\backslash B)$ as required.

(b) Prove $(A \cup B^c)^c = A^c \cap B$. Let U be the universal set, and let $x \in U$. We prove $x \in (A \cup B^c)^c \Longleftrightarrow x \in A^c \cap B$.

$\Longrightarrow$ Let $x \in (A \cup B^c)^c$. Then $x \notin A \cup B^c$, so $x \notin A$ and $x \in B$. Since $x \notin A$, $x \in A^c$ so $x \in A^c \cap B$ as required.

$\Longleftarrow$ Let $x \in A^c \cap B$. Then $x \in A^c$ and $x \in B$, so $x \notin A$ and $x \notin B^c$. Therefore $x \notin A \cup B^c$, and $x \in (A \cup B^c)^c$ as required.

(c) We need to prove 2 things; that the empty set ϕ has no proper subset, and that every nonempty set has a proper subset.

Assume (for a contradiction) that A is a proper subset of ϕ. Then (by the definition of subset) $x \in A \Longrightarrow x \in \phi$. Since ϕ has no elements we conclude that A has no elements. In other words, $A = \phi$. But the definition of proper subset requires that $A \neq \phi$ giving the contradiction we require.

Now assume that we have a set A such that $A \neq \phi$. We show that ϕ is a proper subset of A. To prove that $\phi \subseteq A$ we need to show that $x \in \phi \Longrightarrow x \in A$. This is vacuously true since ϕ

has no elements. Lastly, ϕ is a proper subset because there must be some x such that $x \in A$ (otherwise A would be the empty set).

(Note: the idea of this proof is very simple, but needs to be carefully worded to be a watertight proof.)