Integration using partial fractions

This technique is needed for integrands which are rational functions, that is, they are the quotient of two polynomials. We can sometimes use long division in order to rewrite such an integrand into a sum of functions whose antiderivatives we can easily find.

Recall If p is a polynomial in the variable x, the **degree of *p***, deg(p) is defined to be the highest power of x in $p(x)$.

Examples $3x$ has degree 1, $1-x-x^2$ has degree 2, 5 has degree 0.
We could write this: deg($3x$) = 1, deg($1-x-x^2$) = 2, deg(5) = 0.

Revision of long division.

Example Simplify $\dfrac{x^3-3x^2+x+1}{x-1}$ using long division.

We divide x^3-3x^2+x+1 by $(x-1)$. We are going to decompose the numerator into products of the denominator, where possible. The following whos some steps of the process, beginning by writing

$$(x-1)\overline{\big)x^3-3x^2+x+1}\;.$$

We now ask, how many times will x, the term with the highest power of the variable in the denominator, $x-1$, "go into" x^3 , the term with the highest power of the variable in the numerator, x^3-3x^2+x+1.

Since $x^3 \div x = x^2$, we can say it "goes" x^2 times and write x^2 on the line above.

$$\begin{array}{r} x^2 \\ (x-1)\overline{\big)x^3-3x^2+x+1} \end{array}$$

Now multiply: $x^2(x-1) = x^3-x^2$, write this below, and subtract this product of the denominator from x^3-3x^2+x+1 to give a remainder, a term that has not yet been divided by the denominator.
Be very careful to use brackets here.

$$\begin{array}{r} x^2 \\ (x-1)\overline{\big)x^3-3x^2+x+1} \\ \underline{-(x^3-x^2)} \\ -2x^2+x+1 \end{array}$$

Look next at the term with the highest power of x in the remainder, $-2x^2+x+1$, and ask, how many times will x "go into" $-2x^2$? Since $-2x^2 \div x = -2x$, we can write $-2x$ on the line above.

Now multiply: $-2x(x-1) = -2x^2+2x$, write this below, and subtract this product of the denominator from $-2x^2+x+1$.

The process goes on until we have zero remainder, which must happen in this case as $(x-1)$ is a factor of x^3-3x^2+x+1.

$$\begin{array}{r} x^2-2x-1 \\ (x-1)\overline{\big)x^3-3x^2+x+1} \\ \underline{-(x^3-x^2)} \\ -2x^2+x+1 \\ \underline{-(-2x^2+2x)} \\ -x+1 \\ \underline{-(-x+1)} \\ 0 \end{array}$$

Result: $\dfrac{x^3-3x^2+x+1}{x-1}$ **=** $x^2-2x-1,\ \ x \neq 1$, and it is now easy to find an antiderivative.

Example with remainder Simplify $\dfrac{3x^2-3x+4}{x^2-2x}$ using long division.

$$\begin{array}{r} 3 \\ (x^2-2x)\overline{\big)3x^2-3x+4} \\ -(3x^2-6x) \\ \hline 3x+4 \end{array}$$

Divide first by x^2. The remainder is $3x + 4$: we cannot use long division to divide this by $x^2 - 2x$ because the degree of the denominator (which is 2) is higher than the degree of the numerator (1). $3x + 4$ has not been divided by $x^2 - 2x$.

$$\begin{array}{r} 3 \quad + \dfrac{3x+4}{x^2-2x} \\ (x^2-2x)\overline{\big)3x^2-3x+4} \\ -(3x^2-6x) \\ \hline 3x+4 \end{array}$$

We recognize this by simply writing $+\dfrac{3x+4}{x^2-2x}$ above.

Result: $\dfrac{3x^2-3x+4}{x^2-2x} = 3+\dfrac{3x+4}{x^2-2x}$.

Example $\dfrac{6x^3-3x^2+4x-1}{2x+1} = 3x^2-3x+\dfrac{7}{2}-\dfrac{9}{2(2x+1)}$. Do it yourself.

Partial fractions.
If the denominator of a rational function is not a simple linear or quadratic polynomial, as in $\dfrac{3x^2-3x+4}{x^2-2x} = 3+\dfrac{3x+4}{x^2-2x}$, the result after long division will not necessarily be sums of functions whose antiderivatives we can easily find. The technique of **partial fractions** is a method of **decomposing rational functions**, and is very useful for preparing such functions for integration (and has many other uses also).

Consider, we can easily add $\dfrac{3x}{1+x^2}-\dfrac{2}{1-x}$ by finding a common denominator

$$\frac{3x}{1+x^2}-\frac{2}{1-x} = \frac{3x(1-x)-2(1+x^2)}{(1+x^2)(1-x)} = \frac{3x-5x^2-2}{(1+x^2)(1-x)}.$$

What we would like to do is the **same thing backwards**, because the right hand version is not something we would care to integrate, while the left hand version is perfectly reasonable.

Definition The quadratic polynomial q given by $q(x) = ax^2 + bx + c$ (with **coefficients** $a, b, c \in \mathbb{R}$) is said to be **irreducible** if $b^2 - 4ac < 0$, as it cannot then be rewritten as the product of two linear polynomials with real coefficients.

This is just using the quadratic formula to find that if $b^2 - 4ac < 0$, then the equation $ax^2 + bx + c = 0$ has only complex solutions, and so, by the factor theorem (which says that $p(d) = 0$, where p is a polynomial if, and only if, $(x - d)$ is a factor of p), $ax^2 + bx + c$ has only complex linear factors.

Example x^2+1, x^2+x+1, x^2-x+1 are all irreducible.

Method of partial fraction expansion of rational functions

Given $\dfrac{p_0(x)}{q(x)}$ where p_0 and q are polynomials for which $\deg(p_0) \geq \deg(q)$, we use long division to rewrite the expression. Once we have an expression $\dfrac{p(x)}{q(x)}$ for which $\deg(p) < \deg(q)$, we may rewrite $\dfrac{p(x)}{q(x)}$ as a sum of terms called **partial fractions**, whose antiderivatives are known. In order to do so, we first consider the factors in the denominator. We say the factor $(ax + b)$ or $(ax^2 + bx + c)$ of $q(x)$ is **repeated** n times where $(ax + b)^n$ or $(ax^2 + bx + c)^n$ is a factor of $q(x)$.

Examples: $\dfrac{4x^2+3x+1}{x^2(x+1)}$ has x repeated twice and $x + 1$ repeated once in the denominator.

$\dfrac{x^2-x+2}{(x-1)^3(x^2+x+1)^2}$ has $x - 1$ repeated three times and x^2+x+1 (an irreducible quadratic) repeated twice in the denominator.

Knowing this, we factor the denominator and then write down the partial fraction sum (or expansion), using unknown constants.

The sum of partial fractions includes (see examples below):

- the n terms $\dfrac{A_1}{ax+b} + \dfrac{A_2}{(ax+b)^2} + \ldots + \dfrac{A_n}{(ax+b)^n}$

  for each n times repeated **linear** factor $(ax+b)$ in $q(x)$,
  where the numerators $A_1, A_2, \ldots, A_n$ are **constants**;

- the n terms $\dfrac{B_1x+C_1}{ax^2+bx+c} + \dfrac{B_2x+C_2}{(ax^2+bx+c)^2} + \ldots + \dfrac{B_nx+C_n}{(ax^2+bx+c)^n}$

  for each n times repeated **irreducible quadratic** factor (ax^2+bx+c) in $q(x)$,
  where the numerators $B_1x + C_1, \ldots, B_nx + C_n$ are **linear.**

Examples

- $\dfrac{1}{x^2-3x+2} = \dfrac{1}{(x-1)(x-2)} = \dfrac{A}{x-1} + \dfrac{B}{x-2}$
- $\dfrac{4x^2+3x+1}{x^2(x+1)} = \dfrac{A_1}{x} + \dfrac{A_2}{x^2} + \dfrac{B}{x+1}$
- $\dfrac{3x+2}{x(x^2+1)} = \dfrac{A}{x} + \dfrac{B_1x+C_1}{x^2+1}$
- $\dfrac{x^2-x+2}{(x-1)^3(x^2+x+1)^2} = \dfrac{A_1}{x-1} + \dfrac{A_2}{(x-1)^2} + \dfrac{A_3}{(x-1)^3} + \dfrac{B_1x+C_1}{x^2+x+1} + \dfrac{B_2x+C_2}{(x^2+x+1)^2}$

Example $\displaystyle\int_3^4 \frac{1}{x^2-3x+2}\,dx$.

The integrand is a rational function (quotient of two polynomials) with degree of the numerator less than the degree of the denominator, as 0 < 2. We may use the method of partial fractions to decompose the integrand.

Step 1 Rewrite by factoring the denominator, and make the required assumption:

(1) $$\frac{1}{x^2-3x+2} = \frac{1}{(x-1)(x-2)} = \frac{A}{x-1} + \frac{B}{x-2} \quad ,$$

(a **constant** divided by a **linear term** for each linear term in the original denominator).

Step 2 Find the values of A and B for which (1) is **true for all x**. There are many methods, we will use two of these, and both require us to first multiply both sides of (1) by the common denominator $(x-1)(x-2)$ to get an expression without fractions.

(2) $\quad 1 = A(x-2) + B(x-1)$.

Method 1: the cover up rule. The cover up rule is based on the fact that if (2) is true for all real x, it must be true for any particular x. So we choose a value of x which makes one or more terms on the right in (2) zero and we replace this value in the equation. In this case

Step 3
replacing $x = 2$ in (2) gives $\quad 1 = 0 + B(2-1) = B$
replacing $x = 1$ in (2) gives $\quad 1 = A(1-2) + 0 = -A \Rightarrow A = -1$.

$\Rightarrow$ if (2) is true, then $A = -1$ and $B = 1$

$\Rightarrow$ (1) $\quad \dfrac{1}{x^2-3x+2} = \dfrac{1}{(x-1)(x-2)} = \dfrac{-1}{x-1} + \dfrac{1}{x-2}$

You can always check your result by adding the fractions: of course you should get back the original rational expression.

Now we may integrate:

$$\int_3^4 \frac{1}{x^2-3x+2}\,dx = \int_3^4 \left(\frac{-1}{x-1} + \frac{1}{x-2}\right)dx = \left[-\ln|x-1| + \ln|x-2|\right]_3^4$$

$$= \left[\ln\left|\frac{x-2}{x-1}\right|\right]_3^4 = \ln\left(\frac{2}{3}\right) - \ln\left(\frac{1}{2}\right) = \ln\left(\frac{4}{3}\right).$$

You may find you run out of values for x that give zeros in (2) before you have found all the coefficients. In this case just choose other simple values for x that haven't yet been used. You will often get some simultaneous equations to solve.

We need a further result before we use the second method.
Lemma: If any two polynomials have the same values for all $x \in \mathbb{R}$, then the polynomials are identical and so the coefficients of the corresponding powers of x in the two polynomials are equal.
This means that if $a_0 + a_1x + \ldots + a_nx^n = b_0 + b_1x + \ldots + b_ix^i$ is true for all $x \in \mathbb{R}$, where $i \le n$, then

$(a_0 - b_0) + (a_1 - b_1)x + \ldots + (a_i - b_i)x^i + a_{i+1}x^{i+1} \ldots + a_nx^n = 0$ for all $x \in \mathbb{R}$
and $\quad a_0 = b_0$, $a_1 = b_1$, ..., $a_i = b_i$ and $a_{i+1} = \ldots = a_n = 0$.

Example We will do the last problem again using the second method. The procedure follows the last one up to (2) and further expands the right hand side to rewrite it as the **sum of powers of x**.

(2) $\quad 1 = A(x-2) + B(x-1) \quad \Rightarrow \quad$ (3) $\quad 1 = x(A+B) + (-2A-B)$.

Method 2: equating coefficients .

By the lemma above, the coefficients of the corresponding powers of x in (3) must be equal. As
$1 = x(A+B) + (-2A-B)$, then

Step 3

x^0: $\quad 1 = -2A - B \quad$ where we equate the coefficients of x^0, that is, the constants, each side of the equality

x^1: $\quad 0 = A + B \quad$ where we equate the coefficients of $x^1 = x$. The coefficient of x at left is 0

We get the two equations $A+B=0$ and $-2A-B=1$, and solving this linear system gives us A = –1 and B = 1 as before.

Which method is best where?
The cover up rule is quickest where the denominators in the partial fraction expansion have linear factors. Equating coefficients is usually better where the denominators contain irreducible quadratic factors.

Example $\int \frac{3x+2}{x(x^2+1)}dx$

Step 1 assumption. (1) $\frac{3x+2}{x(x^2+1)} = \frac{A}{x} + \frac{B_1 x + B_2}{x^2+1}$,

with a **constant numerator for the linear term** and a **linear numerator for the irreducible quadratic term**.

Step 2 Multiply this equality through by the left hand side denominator, and because we have an irreducible quadratic term, expand the result as a sum of powers of x in order to use the second method of finding the constants.

$3x+2 = A(x^2+1) + x(B_1 x + B_2)$ and therefore

(2) $3x+2 = x^2(A+B_1) + x(B_2) + (A)$.

Step 3 Equating coefficients in (2):

x^2: $0 = A + B_1$

x^1: $3 = B_2$

x^0: $2 = A$

giving A = 2, B_1 = –2, B_2 = 3,

so that $\frac{3x+2}{x(x^2+1)} = \frac{A}{x} + \frac{B_1 x + B_2}{x^2+1} = \frac{2}{x} + \frac{-2x+3}{x^2+1}$.

Then $\int \frac{3x+2}{x(x^2+1)}dx = \int \left(\frac{2}{x} + \frac{-2x+3}{x^2+1}\right)dx$.

To integrate, we will split this integrand into three parts, where the first is an obvious split. The reason for the second may not be obvious immediately. Look at the denominator of the second term $\frac{-2x+3}{x^2+1}$: we would want to make the substitution $u = x^2 + 1$, which needs a multiple of x to appear in the numerator if it is to work. However only one part of the numerator ($-2x$) is a multiple of x, the other is a constant, so we split the quotient in order to deal with each part separately.

$$\int \left(\frac{2}{x} + \frac{-2x}{x^2+1} + \frac{3}{x^2+1}\right)dx = 2\ln|x| - \ln|x^2+1| + 3\tan^{-1}(x) + c$$

Do this integration yourself, and note that we have used the following list of antiderivatives which you ought to know by heart.

For a, b constant,

$$\int \cos(ax)dx = \frac{\sin(ax)}{a} + c \qquad \int \sin(ax)dx = \frac{-\cos(ax)}{a} + c \qquad \int \sec^2(ax)dx = \frac{\tan(ax)}{a} + c$$

$$\int e^{ax}dx = \frac{e^{ax}}{a} + c \qquad \int \frac{1}{ax+b}dx = \frac{1}{a}\ln|ax+b| + c \qquad \int \frac{f'(x)}{f(x)}dx = \ln|f(x)| + c$$

$$\int \frac{dx}{1+x^2} = \tan^{-1}(x) + c .$$

Repeated factors in the denominator.

Where there are **repeated factors**, that is, powers of factors in the denominator, we must recognise this in our decomposition assumption.

Example Evaluate $\int_1^2 \frac{4x^2+3x+1}{x^2(x+1)}\,dx$.

First check that the degree of the numerator of the integrand is lower than that of the denominator. (If not, long division must be done first.) Then write down the decomposition assumption.

$$\frac{4x^2+3x+1}{x^2(x+1)} = \frac{A}{x}+\frac{B}{x^2}+\frac{C}{x+1}$$

where because x was what we call a **repeated factor** in the denominator at left, we need to include the terms with denominators x^1 and x^2 (each with constant numerators, as x is linear). Read page 3 again. Then, multiplying through by the left hand denominator gives

$$4x^2+3x+1 = Ax(x+1)+B(x+1)+Cx^2 .$$

Using the cover up rule, we look for values of x that give zeroes at right.

x = –1: 4 – 3 + 1 = 2 = C.

x = 0: 1 = B

Having run out of zeroes before finding A, we use any other simple value of x.

x = 1: 4 + 3 + 1 = 8 = 2A + 2B + C = 2A + 2 + 2 giving A = 2.

Then

$$\frac{4x^2+3x+1}{x^2(x+1)} = \frac{2}{x}+\frac{1}{x^2}+\frac{2}{x+1}\text{, so that}$$

$$\int_1^2 \frac{4x^2+3x+1}{x^2(x+1)}\,dx = \int_1^2\left(\frac{2}{x}+\frac{1}{x^2}+\frac{2}{x+1}\right)dx = \left[2\ln|x|-\frac{1}{x}+2\ln|x+1|\right]_1^2$$

$$=\left(2\ln(2)-\frac{1}{2}+2\ln(3)\right)-\left(0-1+2\ln(2)\right) \qquad =\frac{1}{2}+2\ln(3) .$$

Example Write down the assumption for the partial fraction decomposition of $\frac{x^2-x+2}{(x-1)^3\left(x^2+x+1\right)^2}$.

First check that the degree of the numerator (2) is lower than that of the denominator (7). Then note:

- (x – 1) has power 3, we need 3 repeats (until all 3 powers are used) with constants in the numerators as it is a linear term.
- $\left(x^2+x+1\right)$ has power 2, we need 2 repeats with linear terms in the numerators as it is an irreducible quadratic.

$$\frac{x^2-x+2}{(x-1)^3\left(x^2+x+1\right)^2}=\frac{A_1}{x-1}+\frac{A_2}{(x-1)^2}+\frac{A_3}{(x-1)^3}+\frac{B_1x+B_2}{x^2+x+1}+\frac{B_3x+B_4}{\left(x^2+x+1\right)^2} .$$

We would then solve for $A_1, A_2, A_3, B_1, B_2, B_2$ and B_4 (I suggest by the method of equating coefficients). However it is a bit nasty, so don't actually do it. Integrating the result is also nasty.

Example Find a partial fraction expansion of $\frac{10x^2-x+15}{(2x-1)\left(x^2+4\right)}$, and hence find $\int \frac{10x^2-x+15}{(2x-1)\left(x^2+4\right)}\,dx$.

Check degrees.

Assume that $\frac{10x^2-x+15}{(2x-1)\left(x^2+4\right)} = \frac{A}{2x-1}+\frac{Bx+C}{x^2+4}$ multiply by LH denominator

$\Rightarrow \quad 10x^2 - x + 15 = A(x^2+4) + (Bx + C)(2x-1)$ and sort the RHS by powers of x

$\Rightarrow \quad 10x^2 - x + 15 = x^2(A+2B) + x(-B+2C) + (4A - C)$. Equating coefficients:

x^2: $10 = A + 2B$
x^1: $-1 = -B + 2C$
x^0: $15 = 4A - C$

We get a system of linear equations in A, B and C:

$$\begin{bmatrix} 1 & 2 & 0 & 10 \\ 0 & -1 & 2 & -1 \\ 4 & 0 & -1 & 15 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 0 & 10 \\ 0 & -1 & 2 & -1 \\ 0 & -8 & -1 & -25 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 0 & 10 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -17 & -17 \end{bmatrix}$$

giving $C = 1$, $B = 3$, $A = 4$.

Therefore $\dfrac{10x^2 - x + 15}{(2x-1)(x^2+4)} = \dfrac{4}{2x-1} + \dfrac{3x+1}{x^2+4}$, so that

$$\int \frac{10x^2 - x + 15}{(2x-1)(x^2+4)}\,dx = \int \left(\frac{4}{2x-1} + \frac{3x+1}{x^2+4} \right) dx \ .$$

To integrate the result, we need to split the second term, because the x in its numerator makes for a nice substitution, but the constant needs different treatment:

$$\int \left(\frac{4}{2x-1} + \frac{3x+1}{x^2+4} \right) dx = \int \frac{4}{2x-1}\,dx + \int \frac{3x}{x^2+4}\,dx + \int \frac{1}{x^2+4}\,dx = I_1 + I_2 + I_3 .$$

Using one of the shortcuts, $I_1 = \dfrac{4\ln|2x-1|}{2} + c = 2\ln|2x-1| + c$ (= $\ln(2x-1)^2 + c$).

I_2 needs the substitution $u = x^2 + 4 \Rightarrow x\,dx = \dfrac{du}{2} \Rightarrow I_2 = \dfrac{3}{2}\displaystyle\int \frac{du}{u} = \frac{3\ln|u|}{2} + c = \frac{3\ln|x^2+4|}{2} + c$.

As $x^2 + 4$ is always positive, this can be rewritten to $\dfrac{3\ln(x^2+4)}{2} + c$.

I_3 has an integrand whose antiderivative is an inverse tan function. We can rewrite it to

$$I_3 = \int \frac{1}{x^2+4}\,dx = \int \frac{1}{4(\frac{x^2}{4}+1)}\,dx = \frac{1}{4}\int \frac{1}{\left(\frac{x}{2}\right)^2 + 1}\,dx$$

$$= \frac{1}{2}\int \frac{1}{u^2+1}\,du$$

$$= \frac{1}{2}\tan^{-1}(u) + c = \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + c .$$

Now make the substitution $u = \dfrac{x}{2} \Rightarrow 2du = dx$

and use the formula given in the table on page 5.

Therefore $\displaystyle\int \left(\frac{4}{2x-1} + \frac{3x+1}{x^2+4} \right) dx = 2\ln|2x-1| + \frac{3\ln(x^2+4)}{2} + \frac{1}{2}\tan^{-1}\frac{x}{2} + c$.

Reading: ABD §8.5. **Exercises**: ABD p 543 # 1, 5, 13, 23, 27, 37.