SEMITOPOLOGICAL GROUPS, BOUZIAD SPACES AND TOPOLOGICAL GROUPS

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Abstract. A semitopological group (topological group) is a group endowed with a topology for which multiplication is separately continuous (multiplication is jointly continuous and inversion is continuous). In this paper we use topological games to show that many semitopological groups are in fact topological groups.

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1 Introduction

A *semitopological group* (*topological group*) is a group endowed with a topology for which multiplication is separately continuous (multiplication is jointly continuous and inversion is continuous).

Recall that a function $f : X \times Y \to Z$ that maps from a product of topological spaces X and Y into a topological space Z is said to be *jointly continuous* at a point $(x, y) \in X \times Y$ if for each neighbourhood W of $f(x, y)$ there exists a pair of neighbourhood U of x and V of y such that $f(U \times V) \subseteq W$. If f is jointly continuous at each point of $X \times Y$ then we say that f is *jointly continuous on* $X \times Y$. A related but weaker notion of continuity is the following. A function $g : X \times Y \to Z$ that maps from a product of topological spaces X and Y into a topological space Z is said to be *separately continuous on* $X \times Y$ if for each $x_0 \in X$ and $y_0 \in Y$ the functions $y \mapsto g(x_0, y)$ and $x \mapsto g(x, y_0)$ are both continuous on Y and X respectively.

Ever since [24] there has been continued interest in determining topological properties of a semitopological group that are sufficient to ensure that it is a topological group. There have been many significant contributions to this area, see [1–10, 13–15, 19, 20, 22, 24–34] to name but a few. Just about all of these results require the semitopological group to be *regular* (i.e., every closed subset and every point outside of this set, can be separated by disjoint open sets) and *Baire*, (i.e., the intersection of any countable family of dense open sets is dense) and satisfy some additional completeness properties. This paper is no exception.

However, in this paper we also require some additional notions from topology and game theory. We shall say that a subset Y of a topological space (X, τ) is *bounded in* X if for each decreasing sequence of open sets $(U_n)_{n\in\mathbb{N}}$ in X, $\bigcap_{n\in\mathbb{N}} \overline{U_n} \neq \varnothing$, whenever $U_n \cap Y \neq \varnothing$ for each $n \in \mathbb{N}$.

The game that we shall consider involves two players which we will call α and β. The "field/court" that the game is played on is a fixed topological space (X, τ) with a fixed dense subset D. The name of the game is the $\mathcal{G}_B(D)$-*game*. After naming the game we need to describe how to "play" the $\mathcal{G}_B(D)$-game. The player labeled β starts the game every time (life is not always fair). For his/her first move the player β must select a pair (B_1, b_1) consisting of a nonempty open subset $B_1 \subseteq X$ and a point $b_1 \in D$. Next, α gets a turn. For α's first move he/she must select a nonempty open subset A_1 of B_1. This ends the first round of the game. In the second round, β goes first again and selects a pair (B_2, b_2) consisting of a nonempty open subset $B_2 \subseteq A_1$ and a point $b_2 \in A_1 \cap D$. Player α then gets to respond by choosing a nonempty open subset A_2 of B_2. This ends the second round of the game. In general, after α and β have played the first n-rounds of the $\mathcal{G}_B(D)$-game, β will have selected pairs $(B_1, b_1), (B_2, b_2), \ldots, (B_n, b_n)$ consisting of nonempty open sets $B_1, B_2, \ldots, B_n$ and points $b_1, b_2, \ldots, b_n$ in D and α will have selected nonempty open subsets $A_1, A_2, \ldots, A_n$ such that

$$A_n \subseteq B_n \subseteq A_{n-1} \subseteq B_{n-1} \subseteq \cdots \subseteq A_2 \subseteq B_2 \subseteq A_1 \subseteq B_1.$$

and $b_{k+1} \in A_k \cap D$ for all $1 \leq k < n$.

At the start of the $(n+1)$-round of the game, β goes first (again!) and selects a pair (B_{n+1}, b_{n+1}) consisting of a nonempty open subset B_{n+1} of A_n and a point $b_{n+1} \in A_n \cap D$. As with the previous n-rounds, the player α gets to respond to this move by selecting a nonempty open subset A_{n+1} of B_{n+1}. Continuing this procedure indefinitely (i.e., continuing on forever) the players α and β produce an infinite sequence $(A_n, (B_n, b_n))_{n \in \mathbb{N}}$ called a *play* of the $\mathcal{G}_B(D)$-game. A *partial play* $\big((A_k, (B_k, b_k)) : 1 \leq k \leq n\big)$ of the $\mathcal{G}_B(D)$-game consists of the first n-moves of the $\mathcal{G}_B(D)$-game.

As with any game, we need to specify a rule to determine who wins (otherwise, it is a very boring game). We shall declare that α *wins* a play $(A_n, (B_n, b_n))_{n \in \mathbb{N}}$ of the $\mathcal{G}_B(D)$-game if: for each decreasing sequence $(U_n)_{n \in \mathbb{N}}$ of open subsets of X, $\bigcap_{n \in \mathbb{N}} \overline{U_n} \neq \varnothing$, whenever $\{k \in \mathbb{N} : b_k \notin U_n\}$ is finite, for every $n \in \mathbb{N}$.

If α does not win a play of the $\mathcal{G}_B(D)$-game then we declare that β *wins* that play of the $\mathcal{G}_B(D)$-game. So every play is won by either α or β and no play is won by both players.

Note that if α wins a play $(A_n, (B_n, b_n))_{n \in \mathbb{N}}$ of the $\mathcal{G}_B(D)$-game then $\bigcap_{n \in \mathbb{N}} \overline{A_n} \neq \varnothing$.

Continuing further into game theory we need to introduce the notion of a strategy. By a *strategy* t for the player β we mean a *'rule'* that specifies each move of the player β in every possible situation. More precisely, a strategy $t := (t_n : n \in \mathbb{N})$ for β is an inductively defined sequence of $\tau \times D$-valued functions. The domain of t_1 is the sequence of length zero, denoted by $\emptyset$. That is, $\mathrm{Dom}(t_1) = \{\emptyset\}$ and $t_1(\emptyset) \in (\tau \setminus \{\varnothing\}) \times D$. If $t_1, t_2, \ldots, t_k$ have been defined then the domain of t_{k+1} is:

$$\{(A_1, A_2, \ldots, A_k) \in (\tau \setminus \{\varnothing\})^k : (A_1, A_2, \ldots, A_{k-1}) \in \mathrm{Dom}(t_k) \text{ and } A_k \subseteq B_k, \text{ where } (B_k, b_k) := t_k(A_1, A_2, \ldots, A_{k-1})\}.$$

For each $(A_1, A_2, \ldots, A_k) \in \mathrm{Dom}(t_{k+1})$, $t_{k+1}(A_1, A_2, \ldots, A_k) := (B_{k+1}, b_{k+1}) \in (\tau \setminus \{\varnothing\}) \times D$ is defined so that $B_{k+1} \subseteq A_k$ and $b_{k+1} \in A_k \cap D$.

A *partial t-play* is a finite sequence $(A_1, A_2, \ldots, A_{n-1})$ such that $(A_1, A_2, \ldots, A_{n-1}) \in \mathrm{Dom}(t_n)$. A *$t$-play* is an infinite sequence $(A_n)_{n \in \mathbb{N}}$ such that for each $n \in \mathbb{N}$, $(A_1, A_2, \ldots, A_{n-1})$ is a partial t-play.

A strategy $t := (t_n : n \in \mathbb{N})$ for the player β is called a *winning strategy* if each play of the form: $(A_n, t_n(A_1, \ldots, A_{n-1}))_{n \in \mathbb{N}}$ is won by β. We will call a topological space (X, τ) a *Bouziad* space if it is regular and there exists a dense subset D of X such that the player β does **not** have a winning strategy in the $\mathcal{G}_B(D)$-game played on X. It follows from [31, Theorem 1] that each Bouziad space is in fact a Baire space.

In this paper we will also be interested in another closely related game. This game, denoted $\mathcal{G}_{SB}(D)$, is identical to the $\mathcal{G}_B(D)$-game, except for the definition of a win. In the $\mathcal{G}_{SB}(D)$-game we say that α *wins* a play $(A_n, (B_n, b_n))_{n\in\mathbb{N}}$ of the $\mathcal{G}_{SB}(D)$-game if (i) there exists a subspace S of X and a subsequence $(b_{n_k})_{k\in\mathbb{N}}$ of $(b_n)_{n\in\mathbb{N}}$, contained in S, such that for each decreasing sequence $(U_n)_{n\in\mathbb{N}}$ of open subsets of S, $\bigcap_{n\in\mathbb{N}} \overline{U_n}^S \neq \varnothing$, whenever $\{k \in \mathbb{N} : b_{n_k} \notin U_n\}$ is finite, for every $n \in \mathbb{N}$ and (ii) every subspace of $C_p(S)$, that is bounded in $C_p(S)$, has a compact closure. We will call a topological space (X, τ) a *strong Bouziad* space if it is completely regular and there exists a dense subset D of X such that the player β does **not** have a winning strategy in the $\mathcal{G}_{SB}(D)$-game played on X. It is easy to show that every strongly Bouziad space is a Bouziad space, and hence a Baire space. On the other hand, if X is completely regular and has the property that every subset of $C_p(X)$, that is bounded in $C_p(X)$, has a compact closure, then X is a Bouziad space if, and only if, it a strongly Bouziad space.

2 Feeble continuity of multiplication

Lemma 1 *Let $(G, \cdot, \tau)$ be a semitopological group. If (G, τ) is a Bouziad space then for each pair of open neighbourboods U and W of identity element $e \in G$ there exists a nonempty open subset V of U such that $V^{-1} \subseteq \overline{W \cdot W \cdot W}$.*

Proof: Suppose, in order to obtain a contradiction, that there exists a pair of open neighbourhoods U and W of $e \in G$ such that for each nonempty open subset V of U, $V^{-1} \not\subseteq \overline{W \cdot W \cdot W}$. From this it follows that for each nonempty open subset $V$of U and each dense subset D' of V there exists a point $x \in V \cap D'$ such that $x^{-1} \notin \overline{W \cdot W}$, because otherwise,

$$V^{-1} \subseteq (\overline{V \cap D'})^{-1} \subseteq W \cdot (V \cap D')^{-1} \subseteq W \cdot \overline{W \cdot W} \subseteq \overline{W \cdot W \cdot W}.$$

Recall that for any nonempty subset A of a semitopological group $(H, \cdot, \tau)$ and any open neighbourhood W of the identity element $e \in H$, $(\overline{A})^{-1} \subseteq W \cdot A^{-1}$.

Now, let D be any dense subset of G such that β does not have a winning strategy in the $\mathscr{G}_B(D)$-game played on G. We will define a (necessarily non-winning) strategy $t := (t_n : n \in \mathbb{N})$ for β in the $\mathscr{G}_B(D)$-game played on G, but first we set, for notational reasons, $A_0 := U$ and $b_0 := e$.

Step 1. Choose $b_1 \in A_0 \cap D$ so that $(b_0^{-1} \cdot b_1)^{-1} = b_1^{-1} \notin \overline{W \cdot W}$. Then choose U_1 to be any open neighbourhood of e, contained in $U \cap W$, such that $b_1 \cdot \overline{U_1} \subseteq A_0$. Then define $t_1(\emptyset) := (b_1 \cdot U_1, b_1)$.

Now, suppose that b_j, U_j and $t_j(A_1, \ldots, A_{j-1})$ have been defined for each $1 \le j \le n$ so that:

(i) $b_j \in A_{j-1} \cap D$ and $(b_{j-1}^{-1} \cdot b_j)^{-1} \notin \overline{W \cdot W}$;

(ii) U_j is an open neighbourhood of e, contained in $U \cap W$, such that $b_j \cdot \overline{U_j} \subseteq A_{j-1}$;

(iii) $t_j(A_1, \ldots, A_{j-1}) := (b_j \cdot U_j, b_j)$.

Step $n+1$. Suppose that A_n is a nonempty open subset of $b_n \cdot U_n$. That is, suppose that A_n is the n^{th} move of the player α. Choose $b_{n+1} \in A_n \cap D$ so that $(b_n^{-1} \cdot b_{n+1})^{-1} \notin \overline{W \cdot W}$. Note that this is possible since $b_n^{-1} \cdot (A_n \cap D)$ is a dense subset of $b_n^{-1} \cdot A_n$ and

$$b_n^{-1} \cdot A_n \subseteq b_n^{-1} \cdot (b_n \cdot U_n) = U_n \subseteq U.$$

Then choose U_{n+1} to be any neighbourhood of e, contained in $U \cap W$, such that $b_{n+1} \cdot \overline{U_{n+1}} \subseteq A_n$. Finally, define $t_{n+1}(A_1, \ldots, A_n) := (b_{n+1} \cdot U_{n+1}, b_{n+1})$. Note that:

(i) $b_{n+1} \in A_n \cap D$ and $(b_n^{-1} \cdot b_{n+1})^{-1} \notin \overline{W \cdot W}$;

(ii) U_{n+1} is an open neighbourhood of e, contained in $U \cap W$, such that $b_{n+1} \cdot \overline{U_{n+1}} \subseteq A_n$;

(iii) $t_{n+1}(A_1, \ldots, A_n) := (b_{n+1} \cdot U_{n+1}, b_{n+1})$.

This completes the definition of t. Since t is not a winning strategy for β there exists a play $(A_n, t_n(A_1, \ldots, A_{n-1}))_{n \in \mathbb{N}}$ where α wins. Now there exists a $k \in \mathbb{N}$ such that $b_k \in \overline{(\bigcap_{n \in \mathbb{N}} B_n) \cdot W}$, because otherwise,

$$\{b_n : k+1 \leq n\} \subseteq [B_k \setminus \overline{(\textstyle\bigcap_{n \in \mathbb{N}} B_n) \cdot W}] \quad \text{for all } k \in \mathbb{N}.$$

In which case,

$$\varnothing \neq \textstyle\bigcap_{k \in \mathbb{N}} \overline{[B_k \setminus \overline{(\bigcap_{n \in \mathbb{N}} B_n) \cdot W}]} \subseteq G \setminus (\bigcap_{n \in \mathbb{N}} B_n) \cdot W,$$

but on the other hand,

$$\textstyle\bigcap_{k \in \mathbb{N}} \overline{[B_k \setminus \overline{(\bigcap_{n \in \mathbb{N}} B_n) \cdot W}]} \subseteq \bigcap_{n \in \mathbb{N}} \overline{B_n} = \bigcap_{n \in \mathbb{N}} B_n \subseteq (\bigcap_{n \in \mathbb{N}} B_n) \cdot W,$$

which is impossible. Therefore, we may indeed choose $k \in \mathbb{N}$ so that

$$b_k \in \overline{(\textstyle\bigcap_{n \in \mathbb{N}} B_n) \cdot W} \subseteq \overline{A_{k+1} \cdot W} \subseteq \overline{b_{k+1} \cdot U_{k+1} \cdot W} \subseteq \overline{b_{k+1} \cdot W \cdot W} = b_{k+1} \cdot \overline{W \cdot W}.$$

Therefore, $(b_k^{-1} \cdot b_{k+1})^{-1} = b_{k+1}^{-1} \cdot b_k \in \overline{W \cdot W}$. However, this contradicts the way b_{k+1} was chosen. This completes the proof. $\square$

If $f : (X, \tau) \to (Y, \tau')$ is a surjection acting between topological spaces (X, τ) and (Y, τ') then we say that f is *feebly continuous on* X if for each nonempty open subset V of Y, $\text{int}[f^{-1}(V)] \neq \varnothing$, [10,16].

Proposition 1 *Let $(G, \cdot, \tau)$ be a semitopological group. If multiplication, $(h, g) \mapsto h \cdot g$, is feebly continuous on $G \times G$ then for each nonempty open subset U of G and $n \in \mathbb{N}$ there exist a point x in U and an open neighbourhood V of the identity element $e \in G$ such that:*

$$x \cdot \underbrace{V \cdot V \cdot V \cdots V}_{n-\text{times}} \subseteq U \quad \text{and} \quad \underbrace{V \cdot V \cdot V \cdots V}_{n-\text{times}} \cdot x \subseteq U.$$

Proof: The proof of this follows from a simple induction argument and the fact that for each $g \in G$, both $\{g \cdot U : U \text{ is a neighbourhood of } e\}$ and $\{U \cdot g : U \text{ is a neighbourhood of } e\}$ are local bases for τ at the point $g \in G$. $\square$

Lemma 2 *Let $(G, \cdot, \tau)$ be a semitopological group. If (G, τ) is a Bouziad space and the multiplication operation $(h, g) \mapsto h \cdot g$ is feebly continuous on $G \times G$ then the multiplication operation, $(h, g) \mapsto h \cdot g$, is jointly continuous on $G \times G$.*

Proof: Since $(G, \cdot, \tau)$ is a semitopological group it is sufficient to show that multiplication is jointly continuous at (e, e). So, in order to obtain a contradiction, we will assume that multiplication is not jointly continuous at (e, e). Therefore, by the regularity of (G, τ), there exists an open neighbourhood W of e so that for every neighbourhood U of e, $U \cdot U \not\subseteq \overline{W}$. Since (G, τ) is a Bouziad space there exists a dense subset D_G of G such that β does not possess a winning strategy in the $\mathscr{G}_B(D_G)$-game played on G.

We will now inductively define a (necessarily non-winning) strategy $t := (t_n : n \in \mathbb{N})$ for the player β in the $\mathscr{G}_B(D_G)$-game played on G.
Step 1. We may choose a point $x \in A_0 := G$ and an open neighbourhood U of $e \in G$ such that

$$x \cdot U \subseteq x \cdot U \cdot U \cdot U \subseteq \overline{x \cdot U \cdot U \cdot U} \subseteq A_0.$$

Next, we may pick $y, z \in U$ such that $y \cdot z \notin \overline{W}$ (i.e., $y \notin \overline{W} \cdot z^{-1}$ and so $U \backslash (\overline{W} \cdot z^{-1}) \neq \varnothing$). By Proposition 1 we may select a point $y' \in U \backslash (\overline{W} \cdot z^{-1})$ and an open neighbourhood V of e, contained in U, such that

$$V \cdot V \cdot V \cdot V \cdot y' \subseteq U \backslash (\overline{W} \cdot z^{-1}).$$

Then, $(V \cdot V \cdot V \cdot x^{-1}) \cdot (x \cdot V) \cdot y' \cdot z \cap \overline{W} = \varnothing$ and so $(\overline{V \cdot V \cdot V} \cdot x^{-1}) \cdot (x \cdot V) \cdot y' \cdot z \cap W = \varnothing$. By Lemma 1 there exists a nonempty open subset B_1 of $x \cdot V \subseteq x \cdot U \subseteq \overline{x \cdot U \cdot U \cdot U} \subseteq A_0$ such that $(B_1)^{-1} \subseteq \overline{V \cdot V \cdot V} \cdot x^{-1}$. Thus, $(B_1)^{-1} \cdot B_1 \cdot y' \cdot z \cap W = \varnothing$. Choose

$$b_1 \in (B_1 \cdot y' \cdot z) \cap D_G \subseteq B_1 \cdot U \cdot U \subseteq x \cdot V \cdot U \cdot U \subseteq x \cdot U \cdot U \cdot U \subseteq A_0.$$

Then define $t_1(\emptyset) := (B_1, b_1)$ and $U_1 := B_1 \cdot y' \cdot z$. Note that: $(B_1)^{-1} \cdot U_1 \cap W = \varnothing$ — $(*_1)$.

Now suppose that (B_j, b_j), U_j and $t_j(A_1, \ldots, A_{j-1})$ have been defined for each $1 \leq j \leq n$ so that:

(i) U_j is an open subset of A_{j-1} and $b_j \in U_j \cap D_G$;

(ii) $\overline{B_j} \subseteq A_{j-1}$ and $(B_j)^{-1} \cdot U_j \cap W = \varnothing$ — $(*_j)$;

(iii) $t_j(A_1, \ldots, A_{j-1}) := (B_j, b_j)$.

Step $n+1$. Suppose that A_n is a nonempty open subset of B_n. That is, suppose that A_n is the n^{th} move of the player α. By Proposition 1 we may choose a point $x \in A_n$ and an open neighbourhood U of $e \in G$ such that

$$x \cdot U \subseteq x \cdot U \cdot U \cdot U \subseteq \overline{x \cdot U \cdot U \cdot U} \subseteq A_n.$$

Next, we may pick $y, z \in U$ such that $y \cdot z \notin \overline{W}$ (i.e., $y \notin \overline{W} \cdot z^{-1}$ and so $U \backslash (\overline{W} \cdot z^{-1}) \neq \varnothing$). Again by Proposition 1 we may select a point $y' \in U \backslash (\overline{W} \cdot z^{-1})$ and an open neighbourhood V of e, contained in U, such that

$$V \cdot V \cdot V \cdot V \cdot y' \subseteq U \backslash (\overline{W} \cdot z^{-1}).$$

Then, $(V \cdot V \cdot V \cdot x^{-1}) \cdot (x \cdot V) \cdot y' \cdot z \cap \overline{W} = \varnothing$ and so $(\overline{V \cdot V \cdot V} \cdot x^{-1}) \cdot (x \cdot V) \cdot y' \cdot z \cap W = \varnothing$. By Lemma 1 there exists a nonempty open subset B_{n+1} of $x \cdot V \subseteq x \cdot U \subseteq A_n$ such that $(B_{n+1})^{-1} \subseteq \overline{V \cdot V \cdot V} \cdot x^{-1}$. Thus, $(B_{n+1})^{-1} \cdot B_{n+1} \cdot y' \cdot z \cap W = \varnothing$. Choose

$$b_{n+1} \in (B_{n+1} \cdot y' \cdot z) \cap D_G \subseteq B_{n+1} \cdot U \cdot U \subseteq x \cdot V \cdot U \cdot U \subseteq x \cdot U \cdot U \cdot U \subseteq A_n.$$

Then define $t_{n+1}(A_1, \ldots, A_n) := (B_{n+1}, b_{n+1})$ and $U_{n+1} := B_{n+1} \cdot y' \cdot z$.

Note that: $\overline{B_{n+1}} \subseteq A_n$ and $(B_{n+1})^{-1} \cdot U_{n+1} \cap W = \varnothing$ — $(*_{n+1})$.

This completes the definition of t. Since t is not a winning strategy for β there exists a play $(A_n, t_n(A_1, \ldots, A_{n-1}))_{n \in \mathbb{N}}$ where α wins. Note that since $\{b_n : n \geq k\} \subseteq (\bigcup_{n \geq k} U_n)$ for all $k \in \mathbb{N}$,

$$\varnothing \neq \bigcap_{k \in \mathbb{N}} (\overline{\bigcup_{n \geq k} U_n}) \subseteq \bigcap_{k \in \mathbb{N}} \overline{A_{k-1}} = \bigcap_{k \in \mathbb{N}} \overline{A_k} = \bigcap_{k \in \mathbb{N}} \overline{B_k} = \bigcap_{n \in \mathbb{N}} B_n.$$

Let $b_\infty \in \bigcap_{k \in \mathbb{N}} (\overline{\bigcup_{n \geq k} U_n})$. Fix $n \in \mathbb{N}$, then by equation $(*_n)$, $b_\infty^{-1} \cdot U_n \cap W = \varnothing$, or equivalently, $U_n \cap (b_\infty \cdot W) = \varnothing$. Hence, $\bigcap_{k \in \mathbb{N}} (\overline{\bigcup_{n \geq k} U_n}) \cap (b_\infty \cdot W) = \varnothing$. However, this contradicts the fact that $b_\infty \in \bigcap_{k \in \mathbb{N}} (\overline{\bigcup_{n \geq k} U_n})$ and W is an open neighbourhood of e. Hence the multiplication operation on G is jointly continuous. □

If $f : (X, \tau) \to (Y, \tau')$ is a function acting between topological spaces (X, τ) and (Y, τ') and $x \in X$ then we say that f is *quasi-continuous at x* if for each neighbourhood W of $f(x)$ and neighbourhood U of x there exists a nonempty open subset $V \subseteq U$ such that $f(V) \subseteq W$, [18].

Theorem 1 *If $(G, \cdot, \tau)$ is a semitopological group and (G, τ) is a Bouziad space then $(G, \cdot, \tau)$ is a topological group if (and only if) the multiplication operation $(h, g) \mapsto h \cdot g$ is feebly continuous on $G \times G$.*

Proof: From Lemma 2 we know that the multiplication operation on G is jointly continuous. Therefore, by Lemma 1, we see that inversion is quasi-continuous at e. The result now follows from [19, Lemma 4] where it is shown that each semitopological group with jointly continuous multiplication and inversion that is quasi-continuous at the identity element is a topological group. □

Hence in the realm of Bouziad spaces, the problem of determining when a semitopological group is a topological group reduces to the problem of determining when a semitopological group has feebly continuous multiplication.

3 Strong quasi-continuity of multiplication

Let X, Y and Z be topological spaces. We will say that a function $f : X \times Y \to Z$ is *strongly quasi-continuous, with respect to the second variable at $(x, y) \in X \times Y$*, if for each neighbourhood W of $f(x, y)$ and each neighbourhood U of x there exists a nonempty open subset $U' \subseteq U$ and a neighbourhood V' of y such that $f(U' \times V') \subseteq W$, [27].

Remarks 1 *It follows from Proposition 1 that the multiplication operation on a semitopological group $(G, \cdot, \tau)$ is feebly continuous on $G \times G$ if, and only if, it is strongly quasi-continuous, with respect to the second variable, at the point $(e, e) \in G \times G$.*

If (X, τ) is a topological space and $a \in X$ then we shall denote by $\mathcal{N}(a)$ the family of all neighbourhoods of a.

For any point a in a topological space (X, τ) and any dense subset D of X we can consider the following two player topological game, called the *$\mathscr{G}_p(a, D)$-game*. To define this game we must first specify the "rules" and then also specify the definition of a "win".

The moves of the player α are simple. He/she must always select a neighbourhood of the point a. However, the moves of the player β may depend upon the previous move of α. Specifically, for his/her first move β may select any point $b_1 \in D$. For α's first move, as mentioned earlier, α must select a neighbourhood A_1 of a. Now, for β's second move he/she must select a point $b_2 \in A_1 \cap D$. For α's second move he/she is entitled to select any neighbourhood A_2 of a. In general, if α has chosen $A_n \in \mathcal{N}(a)$ as his/her n^{th} move of the $\mathscr{G}_p(a, D)$-game then β is obliged to choose a point $b_{n+1} \in A_n \cap D$. The response of α is then simply to choose any neighbourhood A_{n+1} of a. Continuing in this fashion indefinitely, the players α and β produce a sequence $\big((b_n, A_n)\big)_{n\in\mathbb{N}}$ of ordered pairs with $b_{n+1} \in A_n \cap D \subseteq A_n \in \mathcal{N}(a)$ for all $n \in \mathbb{N}$, called a *play* of the $\mathscr{G}_p(a, D)$-game. A *partial play* $\big((b_k, A_k) : 1 \leq k \leq n\big)$ of the $\mathscr{G}_p(a, D)$-game consists of the first n moves of a play of the $\mathscr{G}_p(a, D)$-game. We shall declare α the *winner* of a play $\big((b_n, A_n)\big)_{n\in\mathbb{N}}$ of the $\mathscr{G}_p(a, D)$-game if: for each decreasing sequence $(U_n)_{n\in\mathbb{N}}$ of open subsets of X, $\bigcap_{n\in\mathbb{N}} \overline{U_n} \neq \varnothing$, whenever $\{k \in \mathbb{N} : b_k \notin U_n\}$ is finite, for every $n \in \mathbb{N}$, otherwise, β is declared the winner.

Note that if $\bigcap_{n\in\mathbb{N}} \overline{\{b_k : k \geq n\}} \neq \varnothing$ then α wins the corresponding play $\big((b_n, A_n)\big)_{n\in\mathbb{N}}$.

A *strategy* for the player α is a rule that specifies his/her moves in every possible situation that can occur. More precisely, a strategy for α is an inductively defined sequence of functions $t := (t_n : n \in \mathbb{N})$. The domain of t_1 is D^1 and for each $(b_1) \in D^1$, $t_1(b_1) \in \mathcal{N}(a)$, i.e., $\big((b_1, t_1(b_1))\big)$ is a partial play. Inductively, if $t_1, t_2, \ldots, t_n$ have been defined then the domain of t_{n+1} is defined to be

$$\{(b_1, b_2, \ldots, b_{n+1}) \in D^{n+1} : (b_1, b_2, \ldots, b_n) \in \mathrm{Dom}(t_n) \quad \text{and} \quad b_{n+1} \in t_n(b_1, b_2, \ldots, b_n) \cap D\}.$$

For each $(b_1, b_2, \ldots, b_{n+1}) \in \mathrm{Dom}(t_{n+1})$, $t_{n+1}(b_1, b_2, \ldots, b_{n+1}) \in \mathcal{N}(a)$. Equivalently, for each $(b_1, b_2, \ldots, b_{n+1}) \in \mathrm{Dom}(t_{n+1})$, $\big((b_k, t_k(b_1, \ldots, b_k)) : 1 \leq k \leq n+1\big)$ is a partial play.

A *partial t-play* is a finite sequence $(b_1, b_2, \ldots, b_n) \in D^n$ such that $(b_1, b_2, \ldots, b_n) \in \mathrm{Dom}(t_n)$ or, equivalently, if $b_{k+1} \in t_k(b_1, b_2, \ldots, b_k) \cap D$ for all $1 \leq k < n$. A *t-play* is an infinite sequence $(b_n)_{n\in\mathbb{N}}$ such that for each $n \in \mathbb{N}$, $(b_1, b_2, \ldots, b_n)$ is a partial t-play.

A strategy $t := (t_n : n \in \mathbb{N})$ for the player α is said to be a *winning strategy* if each play of the form: $(b_n, t_n(b_1, b_2, \ldots, b_n))_{n\in\mathbb{N}}$ is won by α.

We shall call a point a a *nearly q_D^*-point* if the player α has a winning strategy in the $\mathscr{G}_p(a, D)$-game played on X. For more information on topological games, see [11].

Lemma 3 *Suppose that D is a dense subset of a topological space (X, τ) and a is a nearly q_D^*-point of X, then the player α possesses a strategy $s := (s_n : n \in \mathbb{N})$ in the $\mathscr{G}_p(a, D)$-game such that every s-play is bounded in X.*

Proof: Let $t := (t_n : n \in \mathbb{N})$ be a winning strategy for the player α in the $\mathscr{G}_p(a, D)$-game. We shall inductively define a new strategy $s := (s_n : n \in \mathbb{N})$ for the player α.

Step 1. Let (b_1) be a partial s-play, i.e., $b_1 \in D$, define $s_1(b_1) := t_1(b_1)$.

Now, suppose that $s_1, s_2, ..., s_k$ have been defined so that for each $1 \leq i \leq k$:

(i) if $(b_{n_1}, ..., b_{n_m})$ is a subsequence of $(b_1, ..., b_i) \in \mathrm{Dom}(s_i)$, then $(b_{n_1}, ..., b_{n_m}) \in \mathrm{Dom}(t_m)$;

(ii) if $(b_1, ..., b_i) \in \mathrm{Dom}(s_i)$ then

$$s_i(b_1, ..., b_i) = \bigcap \{t_m(b_{n_1}, ..., b_{n_m}) : (b_{n_1}, ..., b_{n_m}) \text{ is a subsequence of } (b_1, ..., b_i)\}.$$

Step $k+1$. Suppose that $(b_1, ..., b_{k+1})$ is a partial s-play, i.e., $(b_1, ..., b_k) \in \text{Dom}(s_k)$ and $b_{k+1} \in s_k(b_1, ..., b_k) \cap D$. Let $(b_{n_1}, ..., b_{n_m})$ be any subsequence of $(b_1, ..., b_{k+1})$ and consider the following three cases:

(i) if $n_m < k+1$, then $(b_{n_1}, ..., b_{n_m})$ is a subsequence of $(b_1, ..., b_k)$ and so by the induction hypothesis $(b_{n_1}, ..., b_{n_m}) \in \text{Dom}(t_m)$;

(ii) if $n_m = k+1$ and $m > 1$, then $(b_{n_1}, ..., b_{n_{m-1}})$ is a subsequence of $(b_1, ..., b_k)$ and so by the induction hypothesis $(b_{n_1}, ..., b_{n_{m-1}}) \in \text{Dom}(t_{m-1})$. On the other hand, $b_{n_m} = b_{k+1} \in s_k(b_1, ..., b_k) \cap D \subseteq t_{m-1}(b_{n_1}, ..., b_{n_{m-1}}) \cap D$ and so $(b_{n_1}, ..., b_{n_m}) \in \text{Dom}(t_m)$;

(iii) if $n_m = k+1$ and $m = 1$, i.e., if $n_1 = k+1$ then $(b_{n_1}) = (b_{k+1}) \in \text{Dom}(t_1)$.

In all three cases, $(b_{n_1}, ..., b_{n_m}) \in \text{Dom}(t_m)$. Define

$$s_{k+1}(b_1, ..., b_{k+1}) := \bigcap \{t_m(b_{n_1}, ..., b_{n_m}) : (b_{n_1}, ..., b_{n_m}) \text{ is a subsequence of } (b_1, ..., b_{k+1})\}.$$

This completes the definition of s. Now suppose that $(b_n)_{n\in\mathbb{N}}$ is an s-play, we need to show that $\{b_n : n \in \mathbb{N}\}$ is bounded in X. To this end, let $(U_n)_{n\in\mathbb{N}}$ be a decreasing sequence of open sets such that $U_n \cap \{b_k : k \in \mathbb{N}\} \neq \varnothing$ for each $n \in \mathbb{N}$. If for some $n \in \mathbb{N}$, $\{k \in \mathbb{N} : b_k \in U_n\}$ is finite then $\bigcap_{n\in\mathbb{N}} \overline{U_n} \neq \varnothing$. So we may suppose that for each $n \in \mathbb{N}$, $\{k \in \mathbb{N} : b_k \in U_n\}$ is infinite. For each $n \in \mathbb{N}$, let $J_n := \{k \in \mathbb{N} : b_k \in U_n\}$. Then $(J_n)_{n\in\mathbb{N}}$ is a decreasing sequence of infinite subsets of $\mathbb{N}$. Hence there exists a strictly increasing sequence $(n_k)_{k\in\mathbb{N}}$ of the natural numbers such that $n_k \in J_k$ for all $k \in \mathbb{N}$.

We claim that $(b_{n_k})_{k\in\mathbb{N}}$ is a t-play. To show this we must show that for each $m \in \mathbb{N}$, $(b_{n_1}, \ldots, b_{n_m}) \in \text{Dom}(t_m)$. However, $(b_{n_1}, \ldots, b_{n_m})$ is a subsequence of $(b_1, b_2, \ldots, b_{n_m}) \in \text{Dom}(s_{n_m})$. Therefore, by the construction of the strategy s, $(b_{n_1}, \ldots, b_{n_m}) \in \text{Dom}(t_m)$. Hence $(b_{n_k})_{k\in\mathbb{N}}$ is a t-play. Now, $|\{k \in \mathbb{N} : b_{n_k} \notin U_n\}| < n$ for each $n \in \mathbb{N}$, therefore, $\bigcap_{n\in\mathbb{N}} \overline{U_n} \neq \varnothing$. This shows that $\{b_k : k \in \mathbb{N}\}$ is bounded in X. □

Variations of the following result are well-known, see [6, 7, 13, 19, 22].

Lemma 4 *Let X be a strongly Bouziad space, Y be a topological space and Z be a completely regular space. If $f : X \times Y \to Z$ is a separately continuous function and D is a dense subset of Y, then for each nearly q_D^*-point $y^* \in Y$ the function f is strongly quasi-continuous, with respect to the second variable, at each point of $X \times \{y^*\}$.*

Proof: Let D_X be any dense subset of X such that β does not have a winning strategy in the $\mathcal{G}_{SB}(D_X)$-game played on X. (Note: such a dense subset is guaranteed by the fact that X is a strong Bouziad space.) We need to show that f is strongly quasi-continuous, with respect to the second variable, at each point $(x^*, y^*) \in X \times \{y^*\}$. So in order to obtain a contradiction let us assume that f is not strongly quasi-continuous, with respect to the second variable, at some point $(x^*, y^*) \in X \times \{y^*\}$. Then there exist open neighbourhoods W of $f(x^*, y^*)$ and U of x^* so that $f(U' \times V') \not\subseteq \overline{W}$ for each nonempty open subset U' of U and each neighbourhood V' of y_0. By the complete regularity of Z there exists a continuous function $g : Z \to [0, 1]$ such that $g(f(x^*, y^*)) = 1$ and $g(Z \setminus W) = \{0\}$. Let $W' := \{z \in Z : g(z) > 3/4\} \subseteq W$. Note that by possibly making U smaller we may assume that $f(x, y^*) \in W'$ for all $x \in U$. We will now inductively define a strategy $t := (t_n : n \in \mathbb{N})$ for the player β in the $\mathcal{G}_{SB}(D_X)$-game played on X. However, we shall first:

(a) let $s := (s_n : n \in \mathbb{N})$ be a strategy for the player α in the $\mathscr{G}_p(y^*, D)$-game such that every s-play is bounded in Y. Note that by Lemma 3 such a strategy exists;

(b) set (for notational reasons) $x_0 := x^*$, $A_0 := U$ and $V_0 := Y$ and let y_0 be any element of D.

Step 1. Select $(x_1, y_1) \in X \times Y$ and an open set V_1 and $t_1(\emptyset)$ so that:

(i) $y^* \in V_1 := \{y \in V_0 \cap s_1(y_0) : f(x_0, y) \in W'\}$;

(ii) $(x_1, y_1) \in (A_0 \cap D_X) \times (V_1 \cap D)$ and $f(x_1, y_1) \notin \overline{W}$;

(iii) $t_1(\emptyset) := (B_1, x_1)$ where, $B_1 := \{x \in A_0 : f(x, y_1) \notin \overline{W}\}$.

Now suppose that (x_j, y_j), V_j and t_j have been defined for each $1 \leq j \leq n$ so that for each $1 \leq j \leq n$

(i) $y^* \in V_j := \{y \in V_{j-1} \cap s_j(y_0, \ldots, y_{j-1}) : f(x_{j-1}, y) \in W'\}$;

(ii) $(x_j, y_j) \in (A_{j-1} \cap D_X) \times (V_j \cap D)$ and $f(x_j, y_j) \notin \overline{W}$;

(iii) $t_j(A_1, \ldots, A_{j-1}) := (B_j, x_j)$, where $B_j := \{x \in A_{j-1} : f(x, y_j) \notin \overline{W}\}$.

Step $n+1$. Suppose that A_n is a nonempty open subset of B_n. That is, suppose that A_n is the n^{th} move of the player α. Select $(x_{n+1}, y_{n+1}) \in X \times Y$ and open set V_{n+1} and $t_{n+1}(A_1, \ldots, A_n)$ so that:

(i) $y^* \in V_{n+1} := \{y \in V_n \cap s_{n+1}(y_0, \ldots, y_n) : f(x_n, y) \in W'\}$;

(ii) $(x_{n+1}, y_{n+1}) \in (A_n \cap D_X) \times (V_{n+1} \cap D)$ and $f(x_{n+1}, y_{n+1}) \notin \overline{W}$;

(iii) $t_{n+1}(A_1, \ldots, A_n) := (B_{n+1}, x_{n+1})$, where $B_{n+1} := \{x \in A_n : f(x, y_{n+1}) \notin \overline{W}\}$.

This completes the definition of $t := (t_n : n \in \mathbb{N})$. Now since t is not a winning strategy for the player β in the $\mathcal{G}_{SB}(D_X)$-game there exists a play $(A_n, t_n(A_1, \ldots, A_{n-1}))_{n\in\mathbb{N}}$ where α wins. Therefore, there exists a subspace $S \subseteq X$ and a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$, contained in S, such that for each decreasing sequence $(U_n)_{n\in\mathbb{N}}$ of open subsets of S, $\bigcap_{n\in\mathbb{N}} \overline{U_n}^S \neq \varnothing$, whenever $\{k \in \mathbb{N} : x_{n_k} \notin U_n\}$ is finite, for every $n \in \mathbb{N}$.

Define $\varphi : Y \to C_p(S)$ - [the continuous real-valued functions defined on S, endowed with the topology of pointwise convergence on S] by,

$$\varphi(y)(s) := (g \circ f)(s, y) \quad \text{for all } s \in S.$$

Then φ is well-defined and continuous on Y. Now, since $(y_n)_{n\in\mathbb{N}}$ is an s-play, $\{y_n : n \in \mathbb{N}\}$ is bounded in Y and so $\{\varphi(y_m) : m \in \mathbb{N}\}$ is bounded in $C_p(S)$. Thus, by assumption, $\overline{\{\varphi(y_m) : m \in \mathbb{N}\}}^{\tau_p}$ is a compact subspace of $C_p(S)$. Hence the sequence $(\varphi(y_n))_{n\in\mathbb{N}}$ has a cluster point $h \in C(S)$. Now, for each fixed $k \in \mathbb{N}$,

$$f(x_{n_k}, y_i) \in f(\{x_{n_k}\} \times V_i) \subseteq f(\{x_{n_k}\} \times V_{n_k+1}) \subseteq W'$$

for all $i > n_k$, since $y_i \in V_i$ for all $i \in \mathbb{N}$. Therefore, $\varphi(y_i)(x_{n_k}) \in (3/4, 1]$ for all $i > n_k$ and so $h(x_{n_k}) \in [3/4, 1] \subseteq (2/3, 1]$ for all $k \in \mathbb{N}$. Since h is continuous, for every $k \in \mathbb{N}$ there exists a relatively open subset U_k of S such that $x_{n_k} \in U_k \subseteq A_{n_k-1}$ and $h(U_k) \subseteq (2/3, 1]$. Hence the set

$$\bigcap_{k\in\mathbb{N}} \overline{\bigcup_{i\geq k} U_i}^S \neq \varnothing.$$

[Here, $\overline{X}^S$ denotes the closure of a subset X of S with respect to the relative topology on S].
Let $x_\infty \in \bigcap_{k\in\mathbb{N}} \overline{\bigcup_{i\geq k} U_i}^S \subseteq S$. Then $h(x_\infty) \in [2/3, 1]$.
On the other hand, if we again fix $k \in \mathbb{N}$ then

$$f(U_i \times \{y_k\}) \subseteq f(A_{n_i-1} \times \{y_k\}) \subseteq f(A_{i-1} \times \{y_k\}) \subseteq f(A_k \times \{y_k\}) \subseteq Z \setminus \overline{W}$$

for all $i > k$. Therefore, $f(\overline{\bigcup_{i>k} U_i}^S \times \{y_k\}) \subseteq Z \setminus W$ for each $k \in \mathbb{N}$ and so $f(x_\infty, y_k) \in Z \setminus W$ for each $k \in \mathbb{N}$; which implies that $h(x_\infty) = 0$. This however, contradicts our earlier conclusion that $h(x_\infty) \in [2/3, 1]$. Hence f is strongly quasi-continuous, with respect to the second variable, at (x^*, y^*). □

Corollary 1 *Suppose that $(G, \cdot, \tau)$ is a semitopological group such that (G, τ) is a strong Bouziad space. Then $(G, \cdot, \tau)$ is a topological group, provided (G, τ) has at least one nearly q_D^*-point, for some dense subset D of G.*

Proof: Since for each $g \in G$, $x \mapsto g \cdot x$, is a homeomorphism on G, we may assume that the identity element $e \in G$ is a nearly q_D^*-point, for some dense subset D of G. Then by Lemma 4 the multiplication operation on G is strongly quasi-continuous, with repect to the second variable, at the point $(e, e) \in G\times G$. Hence, by Remark 1 the multiplication operation on G is feebly continuous. The result now follows from Theorem 1 since every strongly Bouziad space is a Bouziad space. □

If (X, τ) is a topological space and $a \in X$, then we call the point a, a *q-point* if there exists a sequence of neighbourhoods $(U_n)_{n\in\mathbb{N}}$ of a such that every sequence $(x_n)_{n\in\mathbb{N}}$ in X, with $x_n \in U_n$ for all $n \in \mathbb{N}$, has a cluster-point in X.

Corollary 2 *Suppose that $(G, \cdot, \tau)$ is a semitopological group such that (G, τ) is a Bouziad space. Then $(G, \cdot, \tau)$ is a topological group, provided (G, τ) has at least one q-point.*

Proof: It is well-known that if K is a q-space, then every subset of $C_p(K)$, that is bounded in $C_p(K)$, has a compact closure. Hence, by the comment at the end of the introduction to this paper, we see that (G, τ) is a strong Bouziad space. Moreover, it is clear that every q-point of G is a nearly q_G^*-point of G. Hence this result follows from Corollary 1. □

Corollary 3 ([30]) *Suppose that $(G, \cdot, \tau)$ is a semitopological group such that (G, τ) is pseudo-compact. Then $(G, \cdot, \tau)$ is a topological group, provided every subset of $C_p(G)$, that is bounded in $C_p(G)$, has a compact closure.*

Proof: From the comment at the end of the introduction to this paper, it follows that (G, τ) is a strong Bouziad space. Furthermore, since G is pseudo-compact, every point of G is a nearly q_G^*-point of G. So the result follows from Corollary 1. □

Corollary 4 ([23]) *If $(G, \cdot, \tau)$ is a semitopological group such that (G, τ) is homeomorphic to a product of Čech-complete spaces, then $(G, \cdot, \tau)$ is a topological group.*

Proof: It is easy to see that every nearly strongly Baire space (see, [23] for the definition) is a strong Bouziad space and that every nearly q_D-point (see [23] for the definition) is a nearly q_D^*-point. Now, in [23] it is shown that every space that is homeomorphic to a product of Čech-complete spaces is a nearly strongly Baire space with at least one nearly q_D-point. Hence the result follows from Corollary 1. □

Corollary 5 ([19]) *If $(G, \cdot, \tau)$ is a semitopological group such that (G, τ) is a completely regular strongly Baire space, then $(G, \cdot, \tau)$ is a topological group.*

Proof: It is easy to see that every completely regular, strongly Baire space (see [19] for the definition) is a strong Bouziad space and possesses at least one nearly q_D^*-point. The result then follows from Corollary 1. $\square$

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