The geometry of the six quaternionic equiangular lines in $\mathbb{H}^2$

Shayne Waldron

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Abstract

We give a simple presentation of the six quaternionic equiangular lines in $\mathbb{H}^2$ as an orbit of the primitive quaternionic reflection group of order 720 (which is isomorphic to $2 \cdot A_6$, the double cover of A_6). Other orbits of this group are also seen to give optimal spherical designs (packings) of 10, 15 and 20 lines in $\mathbb{H}^2$, with angles $\{\frac{1}{3}, \frac{2}{3}\}$, $\{\frac{1}{4}, \frac{5}{8}\}$ and $\{0, \frac{1}{3}, \frac{2}{3}\}$, respectively. We consider the origins of this reflection group as one of Blichfeldt's "finite collineation groups" for lines in $\mathbb{C}^4$, and general methods for finding nice systems of quaternionic lines.

Key Words: finite tight frames, quaternionic equiangular lines, equi-isoclinic subspaces, quaternionic reflection groups, representations over the quaternions, Frobenius-Schur indicator, projective spherical t-designs, special and absolute bounds on lines, double cover of A_6.

AMS (MOS) Subject Classifications: primary 05B30, 15B33, 20C25, 20G20, 51M05, 51M20, secondary 15B57, 51E99, 51M15, 65D30.

1 Introduction

There has been considerable interest in determining maximal sets of equiangular lines in the Euclidean spaces $\mathbb{R}^d$, $\mathbb{C}^d$ and $\mathbb{H}^d$, as part of the theory of spherical designs, since it began in the 1970's [DGS77]. The existence of real equiangular lines corresponds to the existence of certain classes of strongly regular graphs [Wal09], the existence of a maximal set of d^2 equiangular lines in $\mathbb{C}^d$, i.e., a SIC, is conjectured by Zauner to hold for all dimensions [Zau10], [ACFW18], and for quaternionic lines (see [CKM16]) no explicit examples of maximal sets of quaternionic equiangular lines were known until Et-Taoui [ET20] presented the following (maximal) set of six equiangular lines in $\mathbb{H}^2$ given by the unit vectors

$$v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \ v_2 = \begin{pmatrix} \frac{\sqrt{2}}{\sqrt{5}} \\ \frac{\sqrt{3}}{\sqrt{5}} \end{pmatrix}, \ v_3 = \begin{pmatrix} \frac{\sqrt{2}}{\sqrt{5}} \\ -\frac{\sqrt{3}}{4\sqrt{5}} + \frac{3}{4}i \end{pmatrix}, \ v_4 = \begin{pmatrix} \frac{\sqrt{2}}{\sqrt{5}} \\ -\frac{\sqrt{3}}{4\sqrt{5}} - \frac{1}{4}i + \frac{1}{\sqrt{2}}j \end{pmatrix},$$
$$v_5 = \begin{pmatrix} \frac{\sqrt{2}}{\sqrt{5}} \\ -\frac{\sqrt{3}}{4\sqrt{5}} - \frac{1}{4}i - \frac{1}{2\sqrt{2}}j + \frac{\sqrt{3}}{2\sqrt{2}}k \end{pmatrix}, \ v_6 = \begin{pmatrix} \frac{\sqrt{2}}{\sqrt{5}} \\ -\frac{\sqrt{3}}{4\sqrt{5}} - \frac{1}{4}i - \frac{1}{2\sqrt{2}}j - \frac{\sqrt{3}}{2\sqrt{2}}k \end{pmatrix}, \qquad (1.1)$$

which are said to have "projective symmetry group" A_6. They are unique up to projective unitary equivalence, and were obtained by solving the system of polynomials giving the equiangularity, i.e.,

$$|\langle v_j, v_k \rangle|^2 = \frac{2}{5}, \quad j \neq k, \qquad |\langle v_j, v_j \rangle|^2 = 1,$$

in the variables $z_a, w_a \in \mathbb{C}^2$, where $v_a = z_a + w_a j \in \mathbb{H}^2$, and

$$\langle v, w \rangle := \sum_{j=1}^{d} \overline{v_j} w_j \quad \text{(Euclidean inner product)}. \qquad (1.2)$$

In this paper, we give a simple presentation of the six equiangular lines in $\mathbb{H}^2$, in an attempt to understand quaternionic equiangular lines in general (in the context of maximal sets of real and complex equiangular lines). The main presentation roughly follows our investigation, showing the motivation, definitions required, and technical details, such as computations undertaken in Magma. We summarise the key findings:

- The six equiangular lines are seen to be the orbit of an (irreducible) quaternionic reflection group H_{720} of order 720, which is isomorphic to $2 \cdot A_6$, the double cover of A_6. For the purpose of comparison with (1.1), the corresponding presentation of the lines is given by the equal-norm vectors

$$\begin{pmatrix} \sqrt{2} + \sqrt{10} \\ \sqrt{3} - i + j + \sqrt{3}k \end{pmatrix}, \begin{pmatrix} \sqrt{2} + \sqrt{10} \\ 2i - 2j \end{pmatrix}, \begin{pmatrix} \sqrt{2} + \sqrt{10} \\ -\sqrt{3} - i + j - \sqrt{3}k \end{pmatrix},$$
$$\begin{pmatrix} -\sqrt{3} - i - j - \sqrt{3}k \\ \sqrt{2} + \sqrt{10} \end{pmatrix}, \begin{pmatrix} 2i + 2j \\ \sqrt{2} + \sqrt{10} \end{pmatrix}, \begin{pmatrix} \sqrt{3} - i - j + \sqrt{3}k \\ \sqrt{2} + \sqrt{10} \end{pmatrix}. \qquad (1.3)$$

- The lines are also the orbit (of a vector in $\mathbb{H}^2$) under the action of a subgroup of H_{720} of order 24, which is the complex reflection subgroup with Shephard-Todd number 4. Interestingly, the orbit of another (complex-valued) vector under this subgroup gives the four equiangular lines in $\mathbb{C}^2$.

- The equiangular lines are not roots of the reflection group H_{720}, which partly explains why they have only recently been found. Indeed, the stabiliser of a line is a reflection free subgroup of order 120 which has a faithful action on the line.

- The projective action of $H_{720} \cong 2{\cdot}A_6$ (which has centre the scalar matrices $\{-1,1\}$) on the six lines is that of A_6, i.e., there are exactly two elements $\pm g$ of the reflection group which give any even permutation of the lines.

- The quaternionic reflection groups have been classified, and H_{720} is Cohen's group of type O_2 [Coh80] (page 320). This is said to be Blichfeldt's collineation group (C) for $\mathbb{C}^4$ [Bli17] (page 142). We consider Blichfeldt's groups (A), (C) and (K) in detail. These are neither presented as reflection groups nor as collineation groups for quaternionic lines, but are seen to give nested (irreducible) quaternionic reflection groups which permute various finite sets of quaternionic lines.

- In principal, it is possible to go from the abstract group $2 \cdot A_6$ to its rank 2 quaternionic representation H_{720}, and then the six quaternionic equiangular lines in $\mathbb{H}^2$ (each one of which is fixed by a reducible subgroup of order $120 = \frac{720}{6}$). This provides a general method for constructing nice (highly symmetric) sets of quaternionic lines from abstract groups.

We assume some familiarity with the quaternions $\mathbb{H}$, which are a noncommutative division algebra, and the linear algebra over them [Coh80], [Zha97], [Wal20a], [Voi21]. This can be routine, e.g., matrix groups (don't swap the order of multiplication), to extremely involved, e.g., defining the determinant (it can't reasonably be done [Asl96]). We will provide appropriate commentary when required.

Throughout, we adopt the following conventions. The Euclidean quaternionic space $\mathbb{H}^d$ will be thought of as a right $\mathbb{H}$-vector space (module), so that $\mathbb{H}$-linear maps are applied on left, and we have

$$A(v\alpha) = (Av)\alpha, \qquad A \in M_d(\mathbb{H}),\ v \in \mathbb{H}^d,\ \alpha \in \mathbb{H},$$

and the inner product (1.2) satisfies

$$\langle v\alpha, w\beta\rangle = \overline{\alpha}\langle v, w\rangle\beta, \qquad \alpha, \beta \in \mathbb{H},\ v, w \in \mathbb{H}^d.$$

We use the "complexification"

$$g = A + Bj \in M_d(\mathbb{H}) \quad \Longleftrightarrow \quad [g]_{\mathbb{C}} := \begin{pmatrix} A & -B \\ \overline{B} & \overline{A} \end{pmatrix} \in M_{2d}(\mathbb{C}),$$

$$v = z + wj \in \mathbb{H}^d \quad \Longleftrightarrow \quad [v]_{\mathbb{C}} := \begin{pmatrix} z \\ \overline{w} \end{pmatrix} \in \mathbb{C}^{2d}, \tag{1.4}$$

where $[\cdot]_{\mathbb{C}}$ is $\mathbb{C}$-linear, $[gv]_{\mathbb{C}} = [g]_{\mathbb{C}}[v]_{\mathbb{C}}$, etc.

2 The primitive quaternionic reflection group

The projective symmetry group (see [CW18], [Wal18]) of the six equiangular lines is A_6, which is generated by the permutations

$$a = (12)(34) \quad \text{(order 2)}, \qquad b = (1235)(46) \quad \text{(order 4)}. \tag{2.5}$$

In [Wal20a], it was shown that unitary matrices which give these permutations of the six equiangular lines (1.1) are given by

$$U_a = \begin{pmatrix} \frac{2}{\sqrt{15}}i - \frac{\sqrt{2}}{\sqrt{15}}j & \frac{\sqrt{2}}{\sqrt{5}}i - \frac{1}{\sqrt{5}}j \\ \frac{\sqrt{2}}{\sqrt{5}}i - \frac{1}{\sqrt{5}}j & -\frac{2}{\sqrt{15}}i + \frac{\sqrt{2}}{\sqrt{15}}j \end{pmatrix},$$
$$U_b = \begin{pmatrix} \frac{1}{2\sqrt{5}} + \frac{1}{2\sqrt{3}}i + \frac{3-\sqrt{5}}{2\sqrt{30}}j + \frac{\sqrt{5}+1}{2\sqrt{10}}k & \frac{\sqrt{3}}{2\sqrt{10}} - \frac{1}{2\sqrt{2}}i + \frac{3+\sqrt{5}}{4\sqrt{5}}j - \frac{\sqrt{3}}{5+\sqrt{5}}k \\ \frac{\sqrt{3}}{2\sqrt{10}} + \frac{1}{2\sqrt{2}}i + \frac{3-\sqrt{5}}{4\sqrt{5}}j + \frac{\sqrt{3}}{5-\sqrt{5}}k & -\frac{1}{2\sqrt{5}} + \frac{1}{2\sqrt{3}}i - \frac{3\sqrt{5}+5}{10\sqrt{6}}j + \frac{\sqrt{5}-1}{2\sqrt{10}}k \end{pmatrix}. \tag{2.6}$$

These satisfy

$$U_a^2 = -I, \qquad U_b^4 = -I,$$

and so they do not generate A_6, as a matrix group.

The six equiangular lines of (1.1) and the unitary matrices of (2.6) can be put in the `Magma` computer algebra system as a quaternion algebra over the field $\mathbb{Q}(\zeta)$, $\zeta = \zeta_{120} = e^{\frac{2\pi i}{120}}$, which contains the required roots $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$, e.g.,

```
F:=CyclotomicField(24);
zeta:=RootOfUnity(24);
Q<i,j,k>:=QuaternionAlgebra<F|-1,-1>;
```

Since $i = \zeta^{30} \in \mathbb{Q}(\zeta)$, care must be taken to distinguish (or identify) it with the i provided by `QuaternionAlgebra`. We did this by entering all quaternions in the form $a_1 + a_2 i + a_3 j + a_4 k$, where $a_j \in \mathbb{R} \cap \mathbb{Q}(\zeta)$ and i, j, k are the units for the quaternion algebra. Since Magma could form, but not calculate with, the matrix group (this feature has subsequently been added)

$$H := \langle U_a, U_b \rangle \subset U_2(\mathbb{H}), \tag{2.7}$$

it was constructed as a group of complex matrices via (1.4), from which it was deduced that H is the finite group of order 720 with small group identifier `<720, 409>`. This group is $2 \cdot A_6$, the double cover of A_6, which is the unique (abstract) group with

$$\text{composition series:} \qquad \begin{array}{cl} 2 \cdot A_6 & \\ \mid & A_6 \\ * & \\ \mid & \mathbb{Z}_2 \\ 1 & \end{array}$$

In other words:

- H is a rank 2 faithful irreducible unitary representation of $2 \cdot A_6$ over $\mathbb{H}$, which could be obtained from the complex representations of the abstract group $2 \cdot A_6$.

A **quaternionic reflection** is defined to be a nonidentity unitary map $g \in U_d(\mathbb{H})$ which fixes a subspace of dimension $d-1$ of $\mathbb{H}^d$ (for us these will have finite order, and sometimes this is taken as part the definition), i.e.,

$$\operatorname{rank}(I-g) = d-1.$$

A nonzero vector in the orthogonal complement of the fixed subspace of a reflection (which defines the fixed subspace) is called a **root** of the reflection, and the subspace it gives a **root line**. A (usually finite) group generated by quaternionic reflections is called a **quaternionic reflection group** [Coh80] or a **symplectic reflection group** [BST23] (when given as a complex matrix group). The conjugate $h^{-1}gh$ in the unitary group $U_d(\mathbb{H})$, or $M_d(\mathbb{H})$, of a reflection g is a reflection, since

$$\operatorname{rank}(I - h^{-1}gh) = \operatorname{rank}(h^{-1}(I-g)h) = \operatorname{rank}(I-g).$$

Correspondingly, the reflection groups are classified up to conjugation in $U_d(\mathbb{H})$. By directly observing that H contains reflections, and then calculating the reflection group generated by these reflections, it was determined that

- H is a primitive quaternionic reflection group with 40 reflections, each of order 3.
- The 40 reflections correspond to 20 root lines (a reflection and its inverse have the same root line), which is the maximum possible.
- H is generated by three reflections, but not two.
- The centre of H is the scalar matrices $\pm I$.

Since U_a and U_b permute the six equiangular lines, so does H, with $\pm g$ giving the same even permutation of the lines. In this way, elements of H can be indexed by the permutation of the six equiangular lines that they give (they cover this element of A_6).

Conjugacy classes of A_6

order	representative	length
1	()	1
2	(12)(34)	45
3	(123)(456)	40
3	(123)	40
4	(1234)(56)	90
5	(12345)	72
5	(13452)	72

Conjugacy classes of $2 \cdot A_6$

order	length	elements covered	lines fixed
1	1	()	6
2	1	()	6
3*	40	(123)(456)	none
3	40	(123)	3
4	90	(12)(34)	2
5	72	(12345)	1
5	72	(13452)	1
6	40	(123)(456)	none
6	40	(123)	3
8	90	(1234)(56)	none
8	90	(1234)(56)	none
10	72	(12345)	1
10	72	(13452)	1

* The 40 reflections of order 3

In particular, we observe that

- The 40 reflections form a conjugacy class of elements of order three, corresponding to (123)(456), i.e., they fix no equiangular lines. Thus the equiangular lines are not roots of the reflections, nor of their orthogonal complements.

- The 40 elements of order three which are not reflections form a conjugacy class, corresponding to (123), i.e., they fix three equiangular lines and cycle the remaining three.

The six equiangular lines of (1.1) are the orbit of $v_1 = e_1$ under the action of the reflection group H. Therefore, the first column of each matrix in H is a vector in one of the lines. Any unitary image of these lines is a set of six equiangular lines, which is an orbit of the corresponding conjugate of the group H in $U_2(\mathbb{H})$. We now consider such a presentation of the lines given by a conjugate of H, for which a generating set of reflections takes a simple form.

3 The Blichfeldt generators

The (irreducible) quaternionic reflection groups were classified in Cohen [Coh80]. Cohen classifies the primitive quaternionic reflection groups by whether their complexification is primitive or not (also see [Sch23]). For our group H, the complexifications of the generators U_a and U_b are elements of order 4 and 8 with eigenvalues $i, i, -i, -i$ and $\pm\sqrt{i}, \pm i\sqrt{i}$. It is easily verified that the 1-dimensional eigenspaces for $[U_b]_{\mathbb{C}}$ have trivial intersection with the 2-dimensional ones for $[U_a]_{\mathbb{C}}$, and so the complexification of H is primitive. Cohen gives a list of 16 "exceptional" primitive quaternionic reflection groups with primitive complexifications, in dimensions $1, 2, 3, 4, 5$, of which there are six for $\mathbb{H}^2$, including a unique one of order 720. Therefore

- H is the unique primitive quaternionic reflection group of order 720.

Cohen [Coh80] describes the primitive quaternionic reflection group of order 720 as Blichfeldt's "primitive simple group of collineations in four variables (C) of order 360ϕ". The generators for Blichfeldt's group (C) given in [Bli17] (page 141) are, in the notation of the day:

$$F_1 = (1, 1, \omega, \omega^2), \quad \omega = -\frac{1}{2} + \frac{\sqrt{3}}{2} i,$$

$$F_2: \ x_1 = \frac{1}{\sqrt{3}}(x_1' + \sqrt{2}x_4'), \ x_2 = \frac{1}{\sqrt{3}}(-x_2' + \sqrt{2}x_3'), \ x_3 = \frac{1}{\sqrt{3}}(\sqrt{2}x_2' + x_3'), \ x_4 = \frac{1}{\sqrt{3}}(\sqrt{2}x_1' - x_4'),$$

$$F_3: \ x_1 = \frac{1}{2}(\sqrt{3}x_1' + x_2'), \ x_2 = \frac{1}{2}(x_1' - \sqrt{3}x_2'), \ x_3 = x_4', \ x_4 = x_3',$$

$$F_4: \ x_1 = x_2', \ x_2 = x_1', \ x_3 = -x_4', \ x_4 = -x_3',$$

corresponding to the "substitutions of the alternating group" (permutations) (abc), $(ab)(cd)$, $(ab)(de)$, $(ab)(ef)$, respectively. It is one of the 30 types of "primitive collineation groups in four complex variables", and is simple. Effectively, it is a projective representation of the alternating group A_6 (a simple group of order 360) given as a group of matrices (to be factored by the order ϕ subgroup of scalar matrices). Since, as the name suggests, collineation groups map lines to lines (here in $\mathbb{C}^4$), I initially thought the permutation action was on six lines in $\mathbb{C}^4$, but in fact the action is conjugation on a self normalising index 6 subgroup. We have now seen – over a hundred years later – that it also corresponds to a permutation action on six quaternionic lines.

In modern matrix notation, the above generators are

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \omega & 0 \\ 0 & 0 & 0 & \omega^2 \end{pmatrix}, \quad \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 & \sqrt{2} \\ 0 & -1 & \sqrt{2} & 0 \\ 0 & \sqrt{2} & 1 & 0 \\ \sqrt{2} & 0 & 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

These all have determinant 1, and generate a group G of order 1440, with centre the scalar matrices $\langle i \rangle$, i.e., $\phi = 4$, and $G/Z(G) \cong A_6$. There is a related subgroup (A) of order 60ϕ and a supergroup (K) of order 720ϕ given by generators:

$$(A): \ F_1, F_2, F_3, \qquad (C): \ F_1, F_2, F_3, F_4, \qquad (K): \ F_1, F_2, F_3, F_4, F'',$$

where F'' is given by

$$F'': \ x_1 = x_2', \ x_2 = -x_1', \ x_3 = x_4', \ x_4 = -x_3', \qquad \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

These both have centre $\langle i \rangle$, with $G/Z(G)$ being A_5 and S_6, respectively. Blichfeldt's matrices are not in the form (1.4) for symplectic matrices. The first can be put in this form by conjugation with the permutation matrix $P = [e_1, e_3, e_2, e_4]$ of order 2, i.e.,

$$PF_1P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \omega & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \overline{\omega} \end{pmatrix}, \quad PF_2P = \frac{1}{\sqrt{3}} \begin{pmatrix} -1 & 0 & 0 & -\sqrt{2} \\ 0 & -1 & -\sqrt{2} & 0 \\ 0 & -\sqrt{2} & 1 & 0 \\ -\sqrt{2} & 0 & 0 & 1 \end{pmatrix},$$

but the second does not have the form (1.4). By multiplying F_2, F_3, F_4 by $\pm i$, which does not change the determinant, we obtain matrices which P conjugates to the desired form. The corresponding collineation groups, thus obtained, have centre $\langle -1 \rangle$, and can be viewed as subgroups of $U_2(\mathbb{H})$. Consider the matrices $a_1, \ldots, a_5$ so obtained from

$$PF_1^2P, \quad P(iF_2)P, \quad P(-iF_3)P, \quad P(-iF_4)P, \quad P(-F'')P,$$

i.e.,

$$a_1 = \begin{pmatrix} 1 & 0 \\ 0 & \omega^2 \end{pmatrix}, \ a_2 = \begin{pmatrix} \frac{1}{\sqrt{3}}i & -\frac{\sqrt{2}}{\sqrt{3}}k \\ -\frac{\sqrt{2}}{\sqrt{3}}k & \frac{1}{\sqrt{3}}i \end{pmatrix}, \ a_3 = \begin{pmatrix} \frac{1}{2}k - \frac{\sqrt{3}}{2}i & 0 \\ 0 & k \end{pmatrix}, \ a_4 = \begin{pmatrix} k & 0 \\ 0 & -k \end{pmatrix}, \ a_5 = \begin{pmatrix} j & 0 \\ 0 & j \end{pmatrix}. \tag{3.8}$$

It is easily verified that

$$a_1^3 = I, \quad a_2^2 = a_3^2 = a_4^2 = a_5^2 = -I,$$

and that

$$a_1, \qquad a_1 a_2, \qquad a_2 a_3, \qquad a_3 a_4,$$

are reflections of order 3, so that the groups $\langle a_1, \ldots, a_m \rangle$, $1 \le m \le 4$ are reflection groups, as is $\langle a_1, \ldots, a_5 \rangle$. A simple calculation shows that groups $\langle a_1, a_2, a_3 \rangle$, $\langle a_1, a_2, a_3, a_4 \rangle$, $\langle a_1, a_2, a_3, a_4, a_5 \rangle$ are exactly, i.e., have precisely the same elements, as the quaternionic reflection groups of orders 120, 720, 1440 having a primitive complexification given by the root systems O_1, O_2, O_3 of [Coh80] (Table II).

The irreducible quaternionic reflection group $\langle a_1, a_2 \rangle$ of order 24 does not have a primitive complexification, and hence it can be viewed as a (primitive) complex reflection group in $U_2(\mathbb{C})$. It is instructive to see how this happens, which leads to a final tweak of the generators to make this apparent.

Example 3.1 *Consider the irreducible quaternionic reflection group $G = \langle a_1, a_2 \rangle$. The action of the complexification of the generators on $\mathbb{C}^4$ is given by*

$$[a_1]_{\mathbb{C}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -\frac{1}{2} - \frac{\sqrt{3}}{2}i & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -\frac{1}{2} + \frac{\sqrt{3}}{2}i \end{pmatrix}, \quad [a_2]_{\mathbb{C}} = \frac{i}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 & \sqrt{2} \\ 0 & 1 & \sqrt{2} & 0 \\ 0 & \sqrt{2} & -1 & 0 \\ \sqrt{2} & 0 & 0 & -1 \end{pmatrix},$$

and so we have $\mathbb{C}G$-invariant subspaces

$$V_1 = \operatorname{span}_{\mathbb{C}}\{e_1, e_4\}, \qquad V_2 = \operatorname{span}_{\mathbb{C}}\{e_2, e_3\},$$

of $\mathbb{C}^4$, i.e., the complexification is not irreducible. In other words, the $\mathbb{C}G$-module $\mathbb{H}^2$ (of dimension 4) is not irreducible, as it has $\mathbb{C}G$-invariant subspaces

$$W_1 = \operatorname{span}_{\mathbb{C}}\{e_1, e_2 k\}, \qquad W_2 = \operatorname{span}_{\mathbb{C}}\{e_1 i, e_2 j\},$$

and can be written $\mathbb{H}^2 = W \oplus_{\mathbb{C}} Wj$, where W is either of these. By changing the standard basis of the $\mathbb{H}G$-module $\mathbb{H}^2$ to one for a $\mathbb{C}G$-invariant subspace, we obtain a representation where the matrices have complex entries. For example, take the basis $B = \{e_1 i, e_2 j\}$ for W_1, which has basis map

$$u = [e_1, e_2 k] = \begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix}, \tag{3.9}$$

to obtain

$$[a_1]_B = u^{-1} a_1 u = \begin{pmatrix} 1 & 0 \\ 0 & \omega \end{pmatrix}, \quad [a_2]_B = u^{-1} a_2 u = \begin{pmatrix} \frac{1}{\sqrt{3}}i & \frac{\sqrt{2}}{\sqrt{3}} \\ -\frac{\sqrt{2}}{\sqrt{3}} & -\frac{1}{\sqrt{3}}i \end{pmatrix}, \qquad \omega := -\frac{1}{2} + \frac{\sqrt{3}}{2}i.$$

In view of the above discussion, we take as generators for the Blichfeldt groups, the matrices

$$b_j := u^{-1} a_j u \in U_2(\mathbb{H}), \qquad j = 1, \ldots, 5,$$

where u is given by (3.9), which are given by

$$b_1 = \begin{pmatrix} 1 & 0 \\ 0 & \omega \end{pmatrix},\ b_2 = \begin{pmatrix} \frac{1}{\sqrt{3}} i & \frac{\sqrt{2}}{\sqrt{3}} \\ -\frac{\sqrt{2}}{\sqrt{3}} & -\frac{1}{\sqrt{3}} i \end{pmatrix},\ b_3 = \begin{pmatrix} -\frac{\sqrt{3}}{2} i + \frac{1}{2} k & 0 \\ 0 & k \end{pmatrix},\ b_4 = \begin{pmatrix} k & 0 \\ 0 & -k \end{pmatrix},\ b_5 = \begin{pmatrix} j & 0 \\ 0 & j \end{pmatrix}, \tag{3.10}$$

and define a sequence of nested irreducible quaternionic reflection groups

$$H_{24} = \langle b_1, b_2 \rangle, \quad H_{120} = \langle b_1, b_2, b_3 \rangle, \quad H_{720} = \langle b_1, b_2, b_3, b_4 \rangle, \quad H_{1440} = \langle b_1, b_2, b_3, b_4, b_5 \rangle, \tag{3.11}$$

indexed by their orders. The first of these is the Shephard-Todd complex reflection group number 4. We also let $H_3 = \langle b_1 \rangle$, which is a reducible reflection group of order 3.

4 A nice presentation of the six equiangular lines

Each of the six lines of (1.1) is fixed by a subgroup of H of order 120 (index 6), which is therefore reducible. The corresponding equiangular lines for H_{720} can therefore be found as the orbit of a vector which is fixed by a reducible subgroup of order 120. The group H_{720} has two subgroups of order 120 up to conjugacy, which are isomorphic to $2 \cdot A_5$ (the binary icosahedral group), with the class length being six in both cases. One is H_{120} which is an irreducible reflection group, and the other is reducible and contains no reflections. By taking a reducible subgroup of order 120, and finding a line that it fixes (more detail later), one obtains the following "fiducial" vector

$$w := \begin{pmatrix} \sqrt{2} + \sqrt{10} \\ \sqrt{3} - i + j + \sqrt{3} k \end{pmatrix}, \tag{4.12}$$

whose orbit under H_{720} is six equiangular lines. With the ordering:

$$w, \quad b_1 w, \quad b_1^2 w, \quad b_2 w, \quad b_1 b_2 w, \quad b_1^2 b_2 w,$$

they are those of (1.3), i.e.,

$$\begin{pmatrix} \sqrt{2} + \sqrt{10} \\ \sqrt{3} - i + j + \sqrt{3} k \end{pmatrix},\ \begin{pmatrix} \sqrt{2} + \sqrt{10} \\ 2i - 2j \end{pmatrix},\ \begin{pmatrix} \sqrt{2} + \sqrt{10} \\ -\sqrt{3} - i + j - \sqrt{3} k \end{pmatrix},$$

$$\begin{pmatrix} -\sqrt{3} - i - j - \sqrt{3} k \\ \sqrt{2} + \sqrt{10} \end{pmatrix},\ \begin{pmatrix} 2i + 2j \\ \sqrt{2} + \sqrt{10} \end{pmatrix},\ \begin{pmatrix} \sqrt{3} - i - j + \sqrt{3} k \\ \sqrt{2} + \sqrt{10} \end{pmatrix}.$$

We observe that the six equiangular lines are an orbit of the Shephard-Todd complex reflection group H_{24}, and the absolute value of the ratio of coordinates is the golden ratio. With this ordering, Blichfeldt's generators correspond to the permutations

$$b_1 :\ (123)(456), \quad b_2 :\ (14)(36), \quad b_3 :\ (23)(45), \quad b_4 :\ (13)(46).$$

In particular, b_2 fixes lines 2 and 5. The stabiliser in H_{720} of the line given by the fiducial vector w is generated by b_3 and an element which can have order $3, 5, 6, 10$, e.g., the following matrix of order 5

$$g_2 := \begin{pmatrix} \frac{1}{\sqrt{3}}i & \frac{1}{2\sqrt{6}} + \frac{1}{2\sqrt{2}}i + \frac{1}{2\sqrt{2}}j - \frac{\sqrt{3}}{2\sqrt{2}}k \\ \frac{1}{\sqrt{6}} + \frac{1}{\sqrt{2}}j & -\frac{1}{2} + \frac{1}{2\sqrt{3}}i \end{pmatrix} : \ (24356). \tag{4.13}$$

From the indexing of reflections by elements of A_6, it is immediate that H_{720} cannot be generated by two reflections, but it can be by three, e.g., since we following action on the equiangular lines

$$b_1 : \ (123)(456), \quad b_1 b_2 : \ (156)(234), \quad b_2 b_3 : \ (145)(263), \quad b_3 b_4 : \ (123)(465),$$

it follows that H_{720} is generated by the three reflections

$$b_1 = \begin{pmatrix} 1 & 0 \\ 0 & -\frac{1}{2} + \frac{\sqrt{3}}{2}i \end{pmatrix}, \quad b_1 b_2 = \begin{pmatrix} \frac{1}{\sqrt{3}}i & \frac{\sqrt{2}}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} - \frac{1}{\sqrt{2}i} & \frac{1}{2} + \frac{1}{2\sqrt{3}}i \end{pmatrix}, \quad b_3 b_4 = \begin{pmatrix} -\frac{1}{2} + \frac{\sqrt{3}}{2}j & 0 \\ 0 & 1 \end{pmatrix}. \tag{4.14}$$

We now give an explicit conjugation

$$AHA^{-1} = H_{720}, \qquad A \in M_d(\mathbb{H}),$$

which maps the lines (v_j) of (1.1) to the lines (w_j) of (1.3), i.e.,

$$Av_j = w_j \alpha_j, \qquad \exists \alpha_j \in \mathbb{H}.$$

If $\pm h_a$ and $\pm h_b$ are the elements of H_{720} which give the permutations a and b of (2.5) of the lines given by (v_j), then a suitable A is given by one of each of the equations

$$AU_a A^{-1} = \pm h_a, \quad AU_b A^{-1} = \pm h_b \quad \iff \quad AU_a = \pm h_a A, \quad AU_b = \pm h_b A.$$

The latter presentation gives a homogeneous system of the linear equations in the entries of A, which we were able to solve (for a suitable choice of the $\pm$). In this way, we obtained the matrix

$$A := \begin{pmatrix} -\frac{1}{2} - \frac{\sqrt{5}}{2} + \alpha i - \alpha j + (\frac{1}{2} + \frac{\sqrt{5}}{2})k & -2\beta + i - j + 2\beta k \\ -\gamma + (\frac{\sqrt{2}}{2} + 2 - \frac{\sqrt{2}\sqrt{5}}{2})i & 1 - \frac{3\sqrt{2}}{2} + \sqrt{5} - \frac{\sqrt{2}\sqrt{5}}{2} - \delta i \end{pmatrix}, \tag{4.15}$$

$$\alpha = \frac{\sqrt{2}\sqrt{3}}{3} + \frac{\sqrt{2}\sqrt{3}\sqrt{5}}{3} - \frac{5\sqrt{3}}{6} - \frac{\sqrt{3}\sqrt{5}}{6}, \qquad \beta = \frac{\sqrt{2}\sqrt{3}}{3} - \frac{\sqrt{3}\sqrt{5}}{6},$$
$$\gamma = \frac{\sqrt{2}\sqrt{3}\sqrt{5}}{6} + \frac{\sqrt{2}\sqrt{3}}{2} - \frac{2\sqrt{3}}{3}, \qquad \delta = \frac{\sqrt{2}\sqrt{3}\sqrt{5}}{6} + \frac{\sqrt{3}\sqrt{5}}{3} - \frac{\sqrt{2}\sqrt{3}}{6} + \frac{\sqrt{3}}{3}.$$

The diagonal entries of the scalar matrix A^*A are $c = \frac{20}{3}(4 - \frac{\sqrt{5}}{5}(5\sqrt{2} - 4) - \sqrt{2}) > 0$. Thus $U = \frac{1}{\sqrt{c}}A$ is a unitary matrix which gives the desired conjugation. However, the entries of U are not in the cyclotomic field in which we did our calculations.

The conjugates of the U_a and U_b of (2.6) give the following generators for H_{720}

$$AU_a A^{-1} = \begin{pmatrix} -\frac{1}{2\sqrt{3}}i - \frac{1}{\sqrt{3}}j + \frac{1}{2}k & \frac{1}{2\sqrt{6}} + \frac{1}{2\sqrt{2}}i - \frac{1}{2\sqrt{2}}j - \frac{1}{2\sqrt{6}}k \\ -\frac{1}{2\sqrt{6}} + \frac{1}{2\sqrt{2}}i - \frac{1}{2\sqrt{2}}j - \frac{1}{2\sqrt{6}}k & -\frac{1}{\sqrt{3}}i - \frac{1}{\sqrt{3}}j \end{pmatrix},$$
$$AU_b A^{-1} = \begin{pmatrix} \frac{1}{2} + \frac{1}{\sqrt{3}}i - \frac{1}{2\sqrt{3}}j & \frac{1}{2\sqrt{6}} + \frac{1}{2\sqrt{2}}i + \frac{1}{2\sqrt{2}}j + \frac{1}{2\sqrt{6}}k \\ -\frac{1}{2\sqrt{6}} - \frac{1}{2\sqrt{2}}i - \frac{1}{2\sqrt{2}}j + \frac{1}{2\sqrt{6}}k & -\frac{1}{2} + \frac{1}{2\sqrt{3}}i + \frac{1}{\sqrt{3}}j \end{pmatrix}. \tag{4.16}$$

5 The stabiliser groups of the six equiangular lines

We now consider the action of the stabiliser group of one of the six equiangular lines on the line that it fixes. This action of a group of order 120 on $\mathbb{H}$ is far from trivial (the situation for parabolic subgroups of reflection groups and highly symmetric tight frames [BW13]), in fact it is faithful. We first give some generalities about groups which fix a line, and what their action on the line can be.

Let $G \subset M_d(\mathbb{H})$ be a group, and consider its action on a nonzero vector $v \in \mathbb{H}^d$. Two vectors gv and hv in the orbit give the same line if there is some scalar $\alpha \in \mathbb{H}$ for which

$$gv = hv\alpha \quad \iff \quad (h^{-1}g)v = v\alpha.$$

Since such an α is unique, we use the notation $\alpha_g = \alpha_{g,v}$ for the nonzero scalar with

$$gv = v\alpha_g. \tag{5.17}$$

The **stabiliser** (in G) of the line $L = \text{span}_{\mathbb{H}}\{v\}$ given by a nonzero vector $v \in \mathbb{H}^d$, or the **projective stabiliser** (in G) of the vector v, is defined by

$$G_L = G_v := \{g \in G : gv = v\alpha, \exists \alpha \in \mathbb{H}\},$$

(this depends only on L), and the corresponding set of scalars are denoted by

$$\mathbb{H}^*_{G,v} = \{\alpha_g \in \mathbb{H} : gv = v\alpha_g, g \in G_v\}.$$

In view of (5.17), the matrix representation of $g|_L : L \to L$, the restriction of $g \in G_L$ to L, with respect to the $\mathbb{H}$-basis $[v]$ for L is $[g] := [g|_L]_{[v]} = [\alpha_g]$, and so

$$G_L \to \mathbb{H}^* : g \mapsto \alpha_g, \qquad G_L \to M_1(\mathbb{H}) : g \mapsto [\alpha_g],$$

are group homomorphisms. Here $\mathbb{H}^*$ denotes the group of nonzero quaternions under multiplication. If G_L is unitary, then the images above are unit scalars and unitary matrices, respectively.

Proposition 5.1 *Let $G \subset M_d(\mathbb{H})$ be a group, and $v \in \mathbb{H}^d$ a nonzero vector. Then the projective stabiliser G_v is a subgroup of G, and $\mathbb{H}^*_{G,v}$ is a subgroup of $\mathbb{H}^*$ (being the homomorphic image of $G_v \to \mathbb{H}^* : g \mapsto \alpha_g$), with*

$$G_v = G_{v\beta}, \qquad \mathbb{H}^*_{G,v\beta} = \beta^{-1}\mathbb{H}^*_{G,v}\beta, \qquad \beta \in \mathbb{H}^*.$$

The projective stabilisers of points (lines) on the same G-orbit are conjugate and hence are isomorphic, i.e.,

$$G_{hv} = h^{-1}G_v h, \qquad h \in G.$$

Proof: We have already observed that $\mathbb{H}^*_{G,v}$ is the isomorphic image of $g \mapsto \alpha_g$, or we can argue directly from (5.17) that

$$v\alpha_{gh} = (gh)v = g(hv) = g(v\alpha_h) = (gv)\alpha_h = v\alpha_g\alpha_h \quad \Longrightarrow \quad \alpha_{gh} = \alpha_g\alpha_h.$$

For any $\beta \in \mathbb{H}^*$, we have

$$\mathbb{H}_{G,v\beta} = \{\alpha : gv\beta = v\beta\alpha, \exists\alpha\} = \beta^{-1}\{\beta\alpha\beta^{-1} : gv = v\beta\alpha\beta^{-1}, \exists\alpha\}\beta = \beta^{-1}\mathbb{H}^*_{G,v}\beta,$$

$$G_{hv} = \{g : g(hv) = hv\alpha, \exists\alpha\} = h\{h^{-1}gh : h^{-1}ghv = v\alpha, \exists\alpha\}h^{-1} = hG_vh^{-1}.$$

□

If $G \subset M_d(\mathbb{H})$ is irreducible with $d \geq 2$, e.g., $G = H_{720}$, then G_v is reducible for v nonzero, and so is a proper subgroup of G.

Example 5.1 *For $G = H_{720}$ and a v giving one of the six equiangular lines, the stabiliser subgroup G_v has order 120, and is isomorphic to $2 \cdot A_5$, the double cover of A_5. Since A_5 is simple, G_v has normal subgroups of orders 1, 2 and 120. Since $\alpha_{-I} = -1$, it follows (or by direct computation) that $\mathbb{H}^*_{G,v}$, which is a quotient of G_v by a normal subgroup, is $2 \cdot A_5$. As a consequence, the orbit $(gv)_{g \in H_{720}}$ has 720 distinct vectors lying in six lines.*

The faithful action of the stabiliser group of one of the six equiangular lines is given by a subgroup $\mathbb{H}^*_{G,v}$ of $\mathbb{H}^*$ with order 120. The finite subgroups of $\mathbb{H}^*$, i.e., the reflection subgroups of $U_1(\mathbb{H})$, have been classified by Stringham [Str81] (also see [Coh80], [CS03]).

Lemma 5.1 *The finite subgroups of $\mathbb{H}^*$, and hence of $U_1(\mathbb{H})$, up to conjugation, are*

(i) *the cyclic group $\mathcal{C}_m = \langle e^{\frac{2\pi i}{m}} \rangle$, of order m, $m \geq 1$,*

(ii) *the binary dihedral group $\mathcal{D}_m = \langle \mathcal{C}_{2m}, k \rangle$, of order $4m$, $m \geq 2$,*

(iii) *the binary tetrahedral group $\mathcal{T} = \langle \mathcal{D}_2, \frac{-1+i+j+k}{2} \rangle$, of order 24,*

(iv) *the binary octahedral group $\mathcal{O} = \langle \mathcal{T}, \frac{i-1}{\sqrt{2}} \rangle$, of order 48,*

(v) *the binary icosahedral group $\mathcal{I} = \langle \mathcal{D}_2, \frac{\tau - \tau^{-1} i - j}{2} \rangle$, $\tau = \frac{1}{2}(1+\sqrt{5})$, of order 120.*

None of the groups above are isomorphic, and the nontrivial ones are the reflection groups in $U_1(\mathbb{H})$.

Proof: The list of conjugacy classes of finite subgroups is given in [Coh80] as the above, where $\mathcal{D}_m$, has the index range $m \geq 1$. It is clear that

$$\mathcal{C}_4 = \langle i \rangle, \qquad \mathcal{D}_1 = \langle k \rangle,$$

are conjugate, and hence isomorphic. The $\mathcal{C}_m$ and $\mathcal{D}_m$ on the above list are abelian and nonabelian, respectively, and so are not isomorphic, nor are they isomorphic to the nonabelian groups $\mathcal{T}$, $\mathcal{O}$, $\mathcal{I}$ (which contain $\mathcal{D}_2$), i.e., $\mathcal{T} \not\cong \mathcal{D}_6$, $\mathcal{O} \not\cong \mathcal{D}_{12}$, $\mathcal{I} \not\cong \mathcal{D}_{30}$. □

From the above list, it follows that the stabiliser group $\mathbb{H}^*_{G,v}$ for any one of the six equiangular lines is the binary icosahedral group $\mathcal{I} \cong 2 \cdot A_5$ (or a conjugate of it).

Let H be the stabiliser group of the equiangular line given by the vector w of (4.12), i.e.,

$$H = (H_{720})_w = \langle b_3, g_2 \rangle,$$

where the generators b_3 and g_2 are given by (3.10) and (4.13). We have observed that $W = \mathrm{span}_{\mathbb{H}}\{w\}$ is an irreducible H-submodule of $\mathbb{H}^2$ on which the action of H is faithful. Since H is unitary, it follows that the orthogonal complement $W^\perp$ of W is an irreducible H-submodule of $\mathbb{H}^2$, and so its orbit gives a set of six lines. Let

$$w^\perp := \begin{pmatrix} -\sqrt{3} - i + j + \sqrt{3}k \\ \sqrt{2} + \sqrt{10} \end{pmatrix} \in W^\perp. \tag{5.18}$$

With the ordering: $w^\perp$, $b_1w^\perp$, $b_1^2w^\perp$, $b_2w^\perp$, $b_1b_2w^\perp$, $b_1^2b_2w^\perp$, we have a second set of equiangular lines

$$\begin{pmatrix} -\sqrt{3}-i+j+\sqrt{3}k \\ \sqrt{2}+\sqrt{10} \end{pmatrix}, \begin{pmatrix} 2i-2j \\ \sqrt{2}+\sqrt{10} \end{pmatrix}, \begin{pmatrix} \sqrt{3}-i+j-\sqrt{3}k \\ \sqrt{2}+\sqrt{10} \end{pmatrix},$$
$$\begin{pmatrix} \sqrt{2}+\sqrt{10} \\ \sqrt{3}-i-j-\sqrt{3}k \end{pmatrix}, \begin{pmatrix} \sqrt{2}+\sqrt{10} \\ 2i+2j \end{pmatrix}, \begin{pmatrix} \sqrt{2}+\sqrt{10} \\ -\sqrt{3}-i-j+\sqrt{3}k \end{pmatrix}, \tag{5.19}$$

which are an orbit of H_{720}. To understand the action of H_{720} on the line $W^\perp = \mathrm{span}_{\mathbb{H}}\{w^\perp\}$, we calculate the matrix representation for the basis $B = [w, w^\perp]$, i.e., $[g]_B = B^{-1}gB$, of the generators b_3 (order 4) and g_2 (order 5) for H. They are

$$[b_3]_B = \begin{pmatrix} -\frac{\sqrt{3}}{2}i+k & 0 \\ 0 & k \end{pmatrix}, \quad [g_2]_B = \begin{pmatrix} \frac{\tau^{-1}}{2}+\frac{\tau}{\sqrt{3}}i+\frac{\tau^{-1}}{2\sqrt{3}}j & 0 \\ 0 & -\frac{\tau}{2}+\frac{\tau}{2\sqrt{3}}i-\frac{\tau^{-1}}{2\sqrt{3}}j+\frac{\tau^{-1}}{2}k \end{pmatrix}, \tag{5.20}$$

where $\tau = \frac{1}{2}(1+\sqrt{5})$. It is easily verified that

$$\alpha_{b_3,w^\perp} = -\frac{\sqrt{3}}{2}i+k, \qquad \alpha_{g_2,w^\perp} = \frac{\tau^{-1}}{2}+\frac{\tau}{\sqrt{3}}i+\frac{\tau^{-1}}{2\sqrt{3}}j,$$

generate (a conjugate) of the binary icosahedral group $\mathcal{I}$ (of Lemma 5.1), and so the action of H on $W^\perp$ is faithful. However, W and $W^\perp$ are not isomorphic H-submodules of $\mathbb{H}^2 = W \oplus W^\perp$, since otherwise $\mathbb{H}^2$ would be a homogeneous (isotypic) component for that H-module, and hence the H-orbit of every vector in $\mathbb{H}^2$ would be 1-dimensional, which is not the case. In the interest of more general calculations, we show how this follows from character theory.

Every quaternionic representation of a finite group G as matrices in $M_d(\mathbb{H})$, such as H, corresponds to a complex representation as matrices in $M_{2d}(\mathbb{C})$ via (1.4). The irreducible complex representations $\rho : G \to M_{2d}(\mathbb{C})$ that correspond to quaternionic representations are determined by the Frobenius–Schur indicator $\iota\chi$ of their character χ, i.e.,

$$\iota\chi := \frac{1}{|G|}\sum_{g\in g}\chi(g^2) \in \{-1,0,1\}, \qquad \chi(g) = \mathrm{trace}(\rho(g)),$$

taking the value $\iota\chi = -1$ (see [SS95], [Gan11]). For H (as an abstract group) there are two characters corresponding to quaternionic representations of rank 1, i.e.,

The rank 2 characters of $H \cong 2\cdot A_5$

class size	1	1	20	30	12	12	20	12	12
class order	1	2	3	4	5	5	6	10	10
χ_1	2	-2	-1	0	τ^{-1}	$-\tau$	1	τ	$-\tau^{-1}$
χ_2	2	-2	-1	0	$-\tau$	τ^{-1}	1	$-\tau^{-1}$	τ

There are also characters corresponding to irreducible quaternionic representations of rank 2 and rank 3. In view of (1.4), the values of the character χ of the complexification of a quaternionic representation G are given by

$$\chi(g) = \mathrm{trace}([g]_{\mathbb{C}}) = \mathrm{trace}(A) + \mathrm{trace}(\overline{A}) = 2\,\mathrm{Re}(\mathrm{trace}(g)).$$

Hence, by (5.20), the values of the characters of the representations of H on W and $W^\perp$ for the element g_2 are

$$2\,\mathrm{Re}(\mathrm{trace}(g_2|_W) = 2\,\mathrm{Re}\Big(\frac{\tau^{-1}}{2} + \frac{\tau}{\sqrt{3}}i + \frac{\tau^{-1}}{2\sqrt{3}}j\Big) = \tau^{-1} \qquad 2\,\mathrm{Re}(\mathrm{trace}(g_2|_{W^\perp}) = -\tau,$$

and so these representations are different. We now summarise our calculations.

Theorem 5.1 *Let $G = H_{720} = \langle b_1, b_2, b_3, b_4\rangle \cong 2\cdot A_6$ be the primitive quaternionic reflection group of order 720 given by (3.10), with reducible subgroup $H = \langle b_3, g_2\rangle \cong 2\cdot A_5$ of order 120, where g_2 is given by (4.13), and w and $w^\perp$ be the orthogonal vectors*

$$w = \begin{pmatrix} \sqrt{2}+\sqrt{10} \\ \sqrt{3}-i+j+\sqrt{3}k \end{pmatrix}, \qquad w^\perp = \begin{pmatrix} -\sqrt{3}-i+j+\sqrt{3}k \\ \sqrt{2}+\sqrt{10} \end{pmatrix}. \tag{5.21}$$

Then the G-orbits of w and $w^\perp$ each consist of 720 distinct vectors which lie in set of six equiangular lines (120 vectors in each line), and H fixes the lines through w and $w^\perp$, on which it therefore has a faithful irreducible action. Further, we have the orthogonal decomposition

$$\mathbb{H}^2 = \mathrm{span}_{\mathbb{H}}\{w\} \oplus \mathrm{span}_{\mathbb{H}}\{w^\perp\},$$

of $\mathbb{H}^2$ into non-isomorphic irreducible H-submodules, i.e., the homogeneous (isotypic) components.

The fact that the orthogonal complement of equiangular lines in $\mathbb{H}^2$ gives another set of equiangular lines is an example of a more general phenomenon for $\mathbb{H}^2$ ($d = 2$). We will refer to $|\langle v, w\rangle|^2$ as the **angle** between vectors $v, w \in \mathbb{H}^d$.

Proposition 5.2 *For $v \in \mathbb{H}^2$, let $v^\perp$ be any vector orthogonal to v, with $\|v^\perp\| = \|v\|$, e.g.,*

$$v = \begin{pmatrix} a \\ b \end{pmatrix}, \qquad v^\perp = \begin{pmatrix} -\overline{a}^{-1}\overline{b}\overline{a} \\ \overline{a} \end{pmatrix}, \quad a \neq 0.$$

Then $^\perp$ preserves the angles between lines, i.e.,

$$|\langle v, w\rangle|^2 = |\langle v^\perp, w^\perp\rangle|^2, \qquad v, w \in \mathbb{H}^2.$$

Proof: This is by direct computation. Suppose that $v = (a_1, b_1)$, $w = (a_2, b_2)$, with $a_1, a_2 \neq 0$ (the other cases being trivial). Then

$$\langle v, w\rangle = \overline{a_1}a_2 + \overline{b_1}b_2, \qquad \langle v^\perp, w^\perp\rangle = a_1 b_1 a_1^{-1}\overline{a_2}^{-1}\overline{b_2}\overline{a_2} + a_1\overline{a_2}.$$

Using the identities

$$|a+b|^2 = |a|^2 + |b|^2 + 2\,\mathrm{Re}(a\overline{b}), \qquad \mathrm{Re}(ab) = \mathrm{Re}(ba), \qquad |a|^2 a^{-1} = \overline{a}, \quad a \neq 0,$$

for $a, b \in \mathbb{H}$, we calculate

$$\begin{aligned} |\langle v^\perp, w^\perp\rangle|^2 &= |b_1|^2|b_2|^2 + |a_1|^2|a_2|^2 + 2\,\mathrm{Re}(a_1 b_1 a_1^{-1}\overline{a_2}^{-1}\overline{b_2}\overline{a_2}\cdot a_2\overline{a_1}) \\ &= |a_1|^2|a_2|^2 + |b_1|^2|b_2|^2 + 2\,\mathrm{Re}(\overline{a_1}a_2\cdot\overline{b_2}b_1) \\ &= |\langle v, w\rangle|^2, \end{aligned}$$

as claimed. □

6 A general construction of interesting lines

Our construction of the six equiangular lines involved the basic idea of taking the orbit of a vector/line which is fixed by some (ideally large) subgroup, thereby giving

"a small orbit with high symmetry".

In this way, one can obtain a finite class of "lines with high symmetry", by an appropriate choice of definitions. For example:

- *Let G be a finite group with an irreducible action on $\mathbb{H}^d$.*
- *Choose a maximal reducible subgroup $H \subset G$ (there are finitely many) for which $\mathbb{H}^d$ has at least one irreducible one-dimensional H-submodule $L = \operatorname{span}_{\mathbb{H}}\{v\}$ of multiplicity 1, and consider the $n = |G|/|H|$ lines $\{gL\}_{gH \in G/H}$.*

There are *finitely many* sets of "highly symmetric lines" which can be obtained in this way, from any given finite abstract group G. In our case $d = 2$, and so every reducible subgroup H gives and a pair of orthogonal irreducible one-dimensional H-submodules.

If the action of H on L is trivial, i.e., v is fixed, then one obtains what was called a "highly symmetric tight frame" in [BW13] (these sets of n vectors $\{gv\}_{gH \in G/H}$ were given for $\mathbb{C}^d$). When G is a real, complex or quaternionic reflection group, the corresponding subgroups H giving a highly symmetric tight frame are said to be "parabolic subgroups", and these are known to be reflection groups ([Ste64], [BST23] for the quaternionic case).

If the action of H on L is not trivial, then one obtains the "highly symmetric lines" of [Gan22] (given for $\mathbb{C}^d$, but also for 2-dimensional subspaces, so this includes $\mathbb{H}^d$).

Example 6.1 *The irreducible reflection groups $H_{24} \subset H_{120} \subset H_{720} \subset H_{1440}$ contain reflections of order 3, with H_{1440} also containing (and being generated by) reflections of order 2. A subgroup H of one of these reflection groups has the trivial (identity) action on a 1-dimensional subspace L of $\mathbb{H}^2$ if and only if each element of H acts on $\mathbb{H}^2$ as a reflection with root L. Hence H must be H_3 (up to conjugation), or a subgroup generated by a reflection of order 2 in $H_{1440} \setminus H_{720}$. These subgroups generated by a single reflection are therefore the minimal, maximal and hence only nontrivial parabolic subgroups (see [BST23] Lemma 4.2).*

For $G = H_{720}$, there are 17 reducible subgroups H of order > 3 (up to conjugation), which therefore have a nontrivial action on some line L (and its orthogonal complement). Three of these are maximal.

Example 6.2 *The three maximal reducible subgroups of $G = H_{720}$ are (as abstract groups)* `<120, 5>`, `<48, 28>`, `<36, 7>`*, which correspond to sets of 6, 15, 20 lines, with angles $\{\frac{2}{5}\}$, $\{\frac{1}{4}, \frac{5}{8}\}$, $\{0, \frac{1}{3}, \frac{2}{3}\}$, respectively. Generating (fiducial) vectors for these sets of lines are*

$$\begin{pmatrix} \sqrt{2} + \sqrt{10} \\ 2i - 2j \end{pmatrix}, \qquad \begin{pmatrix} 1 \\ j \end{pmatrix}, \qquad \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

The first is an equiangular vector from (1.3), and the third is a root vector (for b_1).

These lines give optimal spherical designs, as we now explain.

For a fixed $t=1,2,\ldots$, a finite set unit vectors $\{v_1,\ldots,v_n\}$ in $\mathbb{H}^d$, equivalently lines, is said to be a **spherical t-design** (or **spherical (t,t)-design**) if there is equality in

$$\sum_{j=1}^n\sum_{k=1}^n |\langle v_j,v_k\rangle|^{2t} \ge c_t(\mathbb{H}^d)\Bigl(\sum_{\ell=1}^n \|v_\ell\|^{2t}\Bigr)^2, \qquad c_t(\mathbb{H}^d):=\frac{2\cdot 3\cdots(t+1)}{2d(2d+1)\cdots(2d+t-1)}, \tag{6.22}$$

i.e.,

$$\frac{1}{n^2}\sum_{j=1}^n\sum_{k=1}^n |\langle v_j,v_k\rangle|^{2t} = c_t(\mathbb{H}^d) = \frac{2\cdot 3\cdots(t+1)}{2d(2d+1)\cdots(2d+t-1)}. \tag{6.23}$$

The term "spherical design" comes from the fact that these can be viewed as cubature rules for the quaternionic sphere, as in the original presentation, which involved harmonic polynomials, see, e.g., [Hog84]. It is quite technical (see [Wal20b]) to show that this definition is equivalent to the "variational characterisation" (6.23), which is easier to verify, and can be used to find numerical constructions. A spherical $(1,1)$-design is a "tight frame" [Wal20a], which is equivalent to the orthogonal-type expansion

$$f=\sum_j v_j\langle v_j,f\rangle, \qquad \forall f\in\mathbb{H}^d. \tag{6.24}$$

Hoggar [Hog78], [Hog82] gave (special) upper bounds on the number of unit vectors in $\mathbb{H}^d$ with a prescribed (small) angle set A, and (absolute) upper bounds which are independent of the angles A. For spherical t-designs for $\mathbb{R}^d$ and $\mathbb{C}^d$, there are lower bounds, depending on t, which when they are (rarely) attained gives the class of so called "tight t-designs" [BMV04], [RS14]. The corresponding theory for "tight quaternionic spherical t-designs" has yet to be developed (see [CKM16]).

The special bounds and absolute bounds for designs in $\mathbb{H}^d$

angles A	special bound $\nu(A)$	absolute bound
$\{\alpha\}$	$\dfrac{d(1-\alpha)}{1-d\alpha}$	$d(2d-1)$
$\{\alpha,\beta\}$	$\dfrac{d(2d+1)(1-\alpha)(1-\beta)}{3-(2d+1)(\alpha+\beta)+d(2d+1)\alpha\beta}$	$\frac{1}{3}d^2(4d^2-1)$
$\{0,\alpha\}$	$\dfrac{d(2d+1)(1-\alpha)}{3-(2d+1)\alpha}$	$\frac{1}{3}d(4d^2-1)$
$\{0,\alpha,\beta\}$	$\dfrac{d(d+1)(2d+1)(1-\alpha)(1-\beta)}{6-3(d+1)(\alpha+\beta)+(d+1)(2d+1)\alpha\beta}$	$\frac{1}{6}d^2(d+1)(4d^2-1)$

The $\nu(A)$ involving β are subject to the restrictions $\alpha+\beta\le\frac{3}{d+1}$ and $\alpha+\beta\le\frac{8}{2d+3}$, respectively.

The six equiangular lines $\mathbb{H}^2$ at angle $\frac{2}{5}$ satisfy both the special and absolute bounds. Before summarising the results of our calculations, we give some further details.

It is easy to determine whether the n lines given by an orbit of v are a spherical t-design by using (6.23). Indeed, since

$$\langle gv, hv \rangle = \langle v, g^* hv \rangle = \langle v, g^{-1} hv \rangle,$$

if (v_j) is a set of unit vectors giving the lines, then (6.23) becomes

$$\sum_{j=1}^{n} |\langle v_j, w \rangle|^{2t} = c_t(\mathbb{H}^d) \cdot n, \tag{6.25}$$

where w is any vector in any line. It follows from this (with the multiplicities indicated) that the 6 and 15 lines with angles $\{\frac{2}{5}\}$ and $\{\frac{1}{4}, \frac{5}{8}\}$ are spherical $(2,2)$-designs, i.e.,

$$1 + 5 \cdot \left(\frac{2}{5}\right)^2 = \frac{3}{10} \cdot 6, \qquad 1 + 6 \cdot \left(\frac{1}{4}\right)^2 + 8 \cdot \left(\frac{5}{8}\right)^2 = \frac{3}{10} \cdot 15.$$

Moreover, the 20 lines with angles $\{0, \frac{1}{3}, \frac{2}{3}\}$ give a spherical $(3,3)$-design, since

$$1 + 1 \cdot 0 + 9 \cdot \left(\frac{1}{3}\right)^3 + 9 \cdot \left(\frac{2}{3}\right)^3 = \frac{1}{5} \cdot 20.$$

It was shown in [MW19] that higher order real and complex spherical designs could be obtained by taking a union of orbits. This concept extends to quaternionic designs:

Example 6.3 *(Mutually unbiased equiangular lines) Take the union of the two sets of six equiangular lines in $\mathbb{H}^2$ at angle $\frac{2}{5}$ given in Theorem 5.1, i.e., the orbit of the orthogonal vectors w and $w^\perp$. It is easily verified that the angle between any vector in one set of lines and and any of the five from the other set which are not orthogonal to it is $\frac{3}{5}$. It follows from (6.25) that these* 12 *vectors with angles* $\{0, \frac{2}{3}, \frac{3}{5}\}$ *give a spherical* $(3,3)$*-design for $\mathbb{H}^2$, by the calculation*

$$1 + 1 \cdot 0 + 5 \cdot \left(\frac{2}{5}\right)^3 + 5 \cdot \left(\frac{3}{5}\right)^3 = \frac{1}{5} \cdot 12.$$

The above union of two sets of equiangular lines can be viewed as an orbit of H_{1440}, i.e., as highly symmetric lines for H_{1440}.

Example 6.4 *The reflection group $G = H_{1440}$ has five maximal reducible subgroups of orders* $120, 72, 48, 48, 24$ *corresponding to sets of* $12, 20, 30, 30, 60$ *highly symmetric lines. Since the index of H_{720} in G is* 2*, the G-orbit of a set of n lines that are an H_{720}-orbit is either the same set of lines or a set of* $2n$ *lines. In this way, for the* $6, 15, 20$ *lines of Example 6.2, we obtain sets of* $12, 30, 20$ *lines.*

The 12 *lines are those of Example 6.3. Indeed, if $r \in H_{1440}$ is the reflection of order* 2 *given by*

$$r := \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1+k \\ 1-k & 0 \end{pmatrix},$$

then it maps the equiangular lines given by w and $w^\perp$ of (5.21) to each other, in particular

$$rw = w^\perp \alpha, \qquad \alpha = \frac{1-k}{\sqrt{2}}.$$

Further, H_{1440} maps the six line orthogonal line pairs $v\mathbb{H}\cup v^\perp\mathbb{H}$ (crosses) to each other. This gives a permutation representation of H_{1440}, with kernel $\langle -I\rangle$. Hence H_{1440} is $2\cdot S_6$, the double cover of S_6, and elements $\pm g$ in H_{1440} can be indexed by permutations on the six equiangular lines (with even permutations mapping a given set of equiangular lines to itself, and odd permutations mapping it to the set of orthogonal equiangular lines).

The 30 lines obtained from the 15 lines with angles $\{\frac14,\frac58\}$, which is a $(2,2)$-design, give a spherical $(3,3)$-design with angles $\{0,\frac14,\frac38,\frac58,\frac34\}$, via the calculation

$$1+1\cdot 0+6\cdot\left(\frac{1}{4}\right)^3+8\cdot\left(\frac{3}{8}\right)^3+8\cdot\left(\frac{5}{8}\right)^3+6\cdot\left(\frac{3}{4}\right)^3=\frac{1}{5}\cdot 30. \tag{6.26}$$

The list of [Hog82] gives one t-design in $\mathbb{H}^2$ meeting the special (and also absolute) bound i.e., the 10 vectors given by the five mutually unbiased orthonormal bases

$$(1,0),\ (0,1),\ (1,\pm1),\ (1,\pm i),\ (1,\pm j),\ (1,\pm k), \tag{6.27}$$

with angles $\{0,\frac12\}$, which form a spherical $(3,3)$-design.

Our calculations have given three new spherical designs for $\mathbb{H}^2$ that meet the special bound, in addition to the two others known.

The special and absolute bounds for spherical t-designs in $\mathbb{H}^2$

n	A	$\nu(A)$	absolute bound	t	
6	$\{\frac25\}$	6	6	2	Equiangular lines [ET20]
10	$\{0,\frac12\}$	10	10	3	[Hog82] (Example 3)
12	$\{0,\frac25,\frac35\}$	12	30	3	Example 6.3, Example 6.4
15	$\{\frac14,\frac58\}$	15	20	2	Example 6.2
20	$\{0,\frac13,\frac23\}$	20	30	3	Example 6.2

Interesting designs can also be found as highly symmetric lines for nonmaximal reducible subgroups of G, e.g., the six equiangular lines for $G=H_{1440}$. Here is another.

Example 6.5 *Let $G=H_{720}$. This has a nonmaximal reducible subgroup of order 24 (a subgroup of the maximal reducible subgroups of orders 120 and 48), which gives a system of 30 lines with angles $\{0,\frac14,\frac12,\frac34\}$ generated by the fiducial vector*

$$\begin{pmatrix}\sqrt{2}i\\ 1+\sqrt{3}\end{pmatrix}.$$

This gives a spherical $(3,3)$-design, via

$$1+1\cdot 0+8\cdot\left(\frac{1}{4}\right)^3+12\cdot\left(\frac{1}{2}\right)^3+8\cdot\left(\frac{3}{4}\right)^3=\frac{1}{5}\cdot 30.$$

This design is fixed by H_{1440}, and so, in view of (6.26), it gives the second set of 30 lines mentioned in Example 6.4.

We now give details about how the calculations discussed were implemented.

7 Computational details

The calculation of all subgroups of a given group G (up to conjugacy) is a task easily done in Magma using `Subgroups(G)`. The identification of those subgroups that are reducible groups of quaternionic matrices, and hence give systems of lines is a little more involved. Serre's condition for irreducibility of finite groups of real or complex matrices [Ser77] (Theorem 5, Chapter 2)

$$\sum_{g\in G} \operatorname{trace}(g) \operatorname{trace}(g^{-1}) = |G|, \tag{7.28}$$

cannot be applied, or generalised in a routine way. However, it was shown in [Wal20a] that a finite group $G \subset U_d(\mathbb{H})$ is irreducible if and only if every orbit of a nonzero vector is a "tight frame", i.e., is a spherical $(1, 1)$-design. From (6.23), it follows that the orbit of a nonzero vector $x \in \mathbb{H}^d$ is spherical (t, t)-design if and only if

$$p_G^{(t)}(x) := \frac{1}{|G|} \sum_{g\in G} |\langle x, gx\rangle|^{2t} - c_t(\mathbb{H}^d)\langle x, x\rangle^{2t} = 0. \tag{7.29}$$

In particular, for $t = 1$, we have the condition for being irreducible

$$\frac{1}{|G|} \sum_{g\in G} |\langle x, gx\rangle|^2 - \frac{1}{d}\langle x, x\rangle^2 = 0. \tag{7.30}$$

This is easily verified in Magma by using `PolynomialRing` to set up an appropriate polynomial ring with the coordinates of x as variables. In this way, we calculated

$$p^{(1)}_{H_{24}} = 0, \qquad p^{(2)}_{H_{720}} = 0, \qquad p^{(3)}_{H_{1440}} = 0,$$

which gives

- Every H_{24} orbit of a nonzero vector gives a spherical $(1, 1)$-design.
- Every H_{720} orbit of a nonzero vector gives a spherical $(2, 2)$-design.
- Every H_{1440} orbit of a nonzero vector gives a spherical $(3, 3)$-design.

Given a reducible subgroup $G \subset U_d(\mathbb{H})$, our method requires the calculation of any 1-dimensional G-invariant subspaces $x\mathbb{H}$ that may exist, i.e., those nonzero $x \in \mathbb{H}^d$ for which

$$gx = x\alpha_g, \quad \exists \alpha_g \in \mathbb{H}, \qquad \forall g \in G. \tag{7.31}$$

The Cauchy-Schwarz inequality (and equality) extends to $\mathbb{H}^d$ (see [Wal20a]), so that (7.31) holds if and only if there is the equality

$$|\langle gx, x\rangle|^2 = \langle gx, gx\rangle\langle x, x\rangle = \langle x, x\rangle^2, \tag{7.32}$$

and hence the set of $x \in \mathbb{H}^d$ giving 1-dimensional G-invariant subspaces is an algebraic variety.

Lemma 7.1 *Let $G \subset U(\mathbb{H}^d)$ be reducible and $x \in \mathbb{H}^d$ be nonzero. Then the line $x\mathbb{H}$ is fixed by G if and only if*

$$|\langle gx, x\rangle|^2 = \langle x, x\rangle^2, \qquad \forall g \in \mathcal{G}, \tag{7.33}$$

where $\mathcal{G}$ is any generating set for G.

Proof: Use the condition (7.32), and the observation that G-invariance is equivalent to invariance under a generating set for G. □

We denote the real algebraic variety given by the set of solutions $x \in \mathbb{H}^d$ to the system of polynomial equations (7.33) by $\mathcal{V}_1(G)$. For us, the computation of $\mathcal{V}_1(G)$ was not completely straightforward, for the following reasons:

- The system of $|\mathcal{G}|$ polynomial equations (7.33) in the $4d$ real variables given by the $1, i, j, k$ coefficients of the coordinates of $x \in \mathbb{H}^d$ is easily formed in Magma. However, in many cases, it could not be solved there. In these cases, the software system Maple was then used to find a numerical solutions, from which analytic ones could then be deduced.
- For reducible subgroups $H_1 \subset H_2$, we have that $\mathcal{V}_1(H_2) \subset \mathcal{V}_1(H_1)$. Often, when finding an element of $\mathcal{V}_1(H_1)$ for a nonmaximal reducible subgroup H_1 of $G = H_{720}$, it turned out to be in the algebraic variety for a maximal subgroup (Example 6.2). Given that the whole variety was not being calculated, it was hard to form a clear picture of whether this was some quirk of the calculation method or because $\mathcal{V}_1(H_1) \setminus \mathcal{V}_1(H_2)$ might be empty. This is an ongoing investigation.

8 Concluding remarks

We have shown that the unique maximal set of six equiangular lines in $\mathbb{H}^2$ is the orbit of a (quaternionic) reflection group. The same is also true for the maximal sets of equiangular lines in $\mathbb{R}^2$ and $\mathbb{C}^2$. The three equiangular lines in $\mathbb{R}^2$ (the Mercedes-Benz frame) are an orbit of the faithful irreducible action of S_3 on $\mathbb{R}^2$, and the SIC of four equiangular lines in $\mathbb{C}^2$ is an orbit of the complex reflection groups with Shephard-Todd numbers $4, 5, 6, 7$ (see [BW13], Table 1). Indeed, the first of these groups is H_{24}, and the H_{24}-orbit of the vector $e_1 = (1, 0)$ gives the following presentation of the SIC

$$\begin{pmatrix}1\\0\end{pmatrix}, \quad \begin{pmatrix}\frac{1}{\sqrt{3}}i\\ -\frac{\sqrt{2}}{\sqrt{3}}\end{pmatrix}, \quad \begin{pmatrix}\frac{1}{\sqrt{3}}i\\ \frac{1}{\sqrt{6}} - \frac{1}{\sqrt{2}}i\end{pmatrix}, \quad \begin{pmatrix}\frac{1}{\sqrt{3}}i\\ \frac{1}{\sqrt{6}} + \frac{1}{\sqrt{2}}i\end{pmatrix}.$$

This SIC is also the orbit of the discrete Heisenberg group [Wal18], which (in this case $d = 2$) is an irreducible real reflection group of order 8.

Moreover, the Hesse SIC of nine equiangular lines in $\mathbb{C}^3$ is an orbit of the complex reflection groups with Shephard-Todd numbers $25, 26$ ([BW13], Example 11, Table 2).

The above examples notwithstanding, we do not expect that the maximal sets of quaternionic equiangular lines in $\mathbb{H}^d$ come as orbits of quaternionic reflection groups, in general, for the following reasons:

- By considering eigenspaces, any element of a finite group $G \subset U_2(\mathbb{C})$ can be multiplied by a unit scalar, to obtain a reflection of the same order. In this way, the appearance of reflection groups for collineation groups acting on $\mathbb{C}^2$ is incidental, rather than by design. Similar reasoning can be applied to collineation groups acting on $\mathbb{H}^2$.

- The general method for finding maximal sets of equiangular lines in $\mathbb{C}^d$ is as the orbit of a fiducial vector under the action of the Heisenberg group (an irreducible projective representation of $\mathbb{Z}_d^2$). These can be viewed as highly symmetric lines for a larger Clifford group [Wal18]. For $d \geq 3$, the Heisenberg and Clifford groups are not given by reflection groups. For $\mathbb{H}^d$, Cohen's classification [Coh80] gives no infinite families of irreducible quaternionic reflection groups. Thus, to find infinite families of optimal quaternionic equiangular lines in this way, one would like an infinite family of irreducible quaternionic matrix groups, of which none are currently known.

We conclude with a couple of obvious directions for extending number of known maximal sets of quaternionic lines:

- Starting with a given quaternionic reflection group, find the associated sets of highly symmetric lines – hoping for an equiangular set.

- Use the variational characterisation of (6.22), or other methods, to find sets of quaternionic equiangular lines numerically (see [CKM16]), and then deduce the irreducible projective quaternionic representation that they might be given by.

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