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An explicit construction of the unitarily invariant quaternionic polynomial spaces on the sphere

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A B S T R A C T

The decomposition of the polynomials on the quaternionic unit sphere in $\mathbb{H}^d$ into irreducible modules under the action of the quaternionic unitary (symplectic) group and quaternionic scalar multiplication has been studied by several authors. Typically, these abstract decompositions into "quaternionic spherical harmonics" specify the irreducible representations involved and their multiplicities.

The elementary constructive approach taken here gives an orthogonal direct sum of irreducibles, which can be described by some low-dimensional subspaces, to which commuting linear operators L and R are applied. These operators map harmonic polynomials to harmonic polynomials, and zonal polynomials to zonal polynomials. We give explicit formulas for the relevant "zonal polynomials" and describe the symmetries, dimensions, and "complexity" of the subspaces involved.

Possible applications include the construction and analysis of desirable sets of points in quaternionic space, such as equiangular lines, lattices and spherical designs (cubature rules).

1. Introduction

There are several desirable sets of points that have been, and are, studied in real, complex and quaternionic space, which includes equiangular lines [2], [26], spherical designs (cubature rules) [15] and lattices [6]. These are usually classified up to "unitary equivalence", and are often constructed as a group orbit of a "unitary action". These considerations have led to this paper.

We consider the invariant subspaces of harmonic polynomials on quaternionic space $\mathbb{H}^d$ under the natural action of the quaternionic unitary matrices (the symplectic group) and scalar multiplication by quaternions (acting on the other side), and the associated zonal polynomials and reproducing kernels. This question has been considered several times, independently, e.g., [23], [6], [4], [9], [3]. The exact answer given depends on

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the precise definition of the harmonic polynomials, in particular, the field in which they may take values, and the precise group and its action. The devil is in the details.

Here we give an elementary examples driven development of this question, motivated by the more well known real and complex cases, the only partly trivial case of $\mathbb{H}^1$, and our interest in the construction of spherical designs for the quaternionic sphere [28], [27]. This proceeds from certain unambiguous definitions and well known facts (the details). We hope that this illuminates the above literature as it applies, and our results can be used for practical computations. Key aspects of our development include:

- By considering the action of scalar multiplication by quaternions on polynomials $\mathbb{H}^d \to \mathbb{C}$, we are naturally led to the operators L and R. The operator R appears in [5] as $\varepsilon^\dagger$, and implicitly in the development of the irreducible representations of the multiplicative group $\mathbb{H}^*$ given in [12].
- There is a natural correspondence between results for homogeneous polynomials and for harmonic polynomials (given by the Fisher decomposition). Ultimately, we are primarily interested in irreducible representations of harmonic polynomials. Sometimes we start with the homogeneous polynomials, as these have natural inner products and explicit bases (of monomials).
- We refer to [21], [20] for some technical calculations. We often give explicit examples, e.g., the operator R in one dimension, or a zonal polynomial with pole $e_1 = (1, 0, \ldots 0)$, to convey the basic ideas behind the results.

2. The devil is in the details

We assume basic familiarity with the quaternions $\mathbb{H}$, with the basis elements $1, i, j, k$. The noncommutative multiplication requires subtle modifications to the associated linear algebra (see [7], [28]). Of particular use is the "commutativity" formula

$$jz = \overline{z}j, \qquad z \in \mathbb{C}. \tag{2.1}$$

We will consider polynomials on real, complex and quaternionic space $\mathbb{R}^d$, $\mathbb{C}^d$ and $\mathbb{H}^d$. With the **Euclidean inner product**

$$\langle v, w\rangle := v^* w = \sum_j \overline{v_j} w_j, \qquad v, w \in \mathbb{F}^d, \quad \mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \tag{2.2}$$

where $\overline{q}$ is the conjugate on $\mathbb{H}$ (and hence $\mathbb{R}$ and $\mathbb{C}$). In the sum above, j is an index, rather than the quaternion j, for which we also use the same symbol (this is commonly done). We will use $\mathbb{F}$ and $\mathbb{K}$ to stand for either of $\mathbb{R}, \mathbb{C}, \mathbb{H}$, independently (9 cases in all for maps $\mathbb{F}^d \to \mathbb{K}$). Given our choice (2.2), it is natural to then treat $\mathbb{H}^d$ as a right $\mathbb{H}$-module, with the $\mathbb{H}$-linear maps L acting on the left, i.e.,

$$\langle v\alpha, w\beta\rangle = \overline{\alpha}\langle v, w\rangle\beta, \qquad (Lv)\alpha = L(v\alpha), \tag{2.3}$$

where $v, w \in \mathbb{H}^d$, $\alpha, \beta \in \mathbb{H}$, and in turn, to make the identification (2.8).

There are natural identifications of $\mathbb{F}^d$ with $\mathbb{R}^{md}$, where $m := \dim_{\mathbb{R}}(\mathbb{F})$, given by the Cayley-Dickson construction of $\mathbb{C}$ and $\mathbb{H}$ from $\mathbb{R}$, e.g., with $(i_1, i_2, i_3, i_4) := (1, i, j, k)$, we have (the $\mathbb{R}$-linear map) $[\cdot]_{\mathbb{R}^{md}} : \mathbb{F}^d \to \mathbb{R}^{md}$ given by

$$[x_1 + i_2 x_2 + \cdots + i_m x_m]_{\mathbb{R}^{md}} = (x_1, \ldots, x_m), \qquad x_1, \ldots, x_m \in \mathbb{R}^d. \tag{2.4}$$

We say $f : \mathbb{F}^d \to \mathbb{R}$ is a **polynomial** if $f([\cdot]_{\mathbb{R}^{md}}^{-1}) : \mathbb{R}^{md} \to \mathbb{R}$ is a polynomial (of md real variables). In this way, we can define **homogeneous** and (**homogeneous**) **harmonic** polynomials $f : \mathbb{F}^d \to \mathbb{R}$ of **degree** k.

These polynomials are real-valued, and naturally form $\mathbb{R}$-vector spaces, which we denote by $\mathrm{Hom}_k(\mathbb{F}^d,\mathbb{R})$ and $\mathrm{Harm}_k(\mathbb{F}^d,\mathbb{R})$.

There is a purely algebraic way to make a finite-dimensional real-vector space into a complex-vector space, and into a (left or right) $\mathbb{H}$-vector space ($\mathbb{H}$-module), by formally multiplying by complex and quaternion scalars. In this way, we define the $\mathbb{K}$-valued $\mathbb{K}$-vector spaces of homogeneous and harmonic polynomials $\mathbb{F}^d \to \mathbb{K}$, which we denote by $\mathrm{Hom}_k(\mathbb{F}^d,\mathbb{K})$ and $\mathrm{Harm}_k(\mathbb{F}^d,\mathbb{K})$. Clearly, with $r = \dim_{\mathbb{R}}(\mathbb{K})$, we have

$$f = f_1 + f_2 i_2 + \cdots + f_r i_r \in \mathrm{Harm}_k(\mathbb{F}^d,\mathbb{K}) \quad \iff \quad f_1,\ldots,f_r \in \mathrm{Harm}_k(\mathbb{F}^d,\mathbb{R}),$$

and similarly for $\mathrm{Hom}_k(\mathbb{F}^d,\mathbb{K})$. Such a $\mathbb{K}$-vector spaces can also be viewed as a $\mathbb{L}$-vector spaces, where $\mathbb{L} \in \{\mathbb{R},\mathbb{C},\mathbb{H}\}$ and $\mathbb{L} \subset \mathbb{K}$. We thereby have (from the real case) the following dimension formulas

$$\begin{aligned}
\dim_{\mathbb{L}}(\mathrm{Hom}_k(\mathbb{F}^d,\mathbb{K})) &= \binom{k+md-1}{md-1}\dim_{\mathbb{L}}(\mathbb{K}),\\
\dim_{\mathbb{L}}(\mathrm{Harm}_k(\mathbb{F}^d,\mathbb{K})) &= \left\{\binom{k+md-1}{md-1} - \binom{k+md-3}{md-1}\right\}\dim_{\mathbb{L}}(\mathbb{K}), \quad md \neq 1\\
&= (2k+md-2)\frac{(k+md-3)!}{(md-2)!k!}\dim_{\mathbb{L}}(\mathbb{K}), \quad k+md-3 \geq 0.
\end{aligned} \tag{2.5}$$

For polynomials $f : \mathbb{F}^d \to \mathbb{K}$, we can define an action of a group G from its action on $\mathbb{F}^d$ via

$$(g \cdot f)(x) := f(g^{-1} \cdot x), \qquad x \in \mathbb{F}^d, \tag{2.6}$$

provided that $[g]_{\mathbb{R}^{md}} : \mathbb{R}^{md} \to \mathbb{R}^{md} : [x]_{\mathbb{R}^{md}} \mapsto [g \cdot x]_{\mathbb{R}^{md}}$ is $\mathbb{R}$-linear. Such a group action preserves the harmonic polynomials of degree k provided that $[g]_{\mathbb{R}^{md}}$ is orthogonal. Since we are only interested in the invariant polynomial subspaces under such an action, it makes no essential difference if we take a left or right action. We say that a nonzero $\mathbb{K}$-subspace V of harmonic polynomials $\mathbb{F}^d \to \mathbb{K}$ is **irreducible** (under the action of G) if its only G-invariant subspaces are V and $\{0\}$, i.e., for every nonzero $f \in V$, we have $\mathrm{span}_{\mathbb{K}}\{g \cdot f : g \in G\} = V$.

We are primarily concerned with polynomials restricted to the (unit) **sphere**

$$\mathbb{S} := \{x \in \mathbb{F}^d : \|[x]_{\mathbb{R}^{md}}\| = 1\}.$$

Hence the linear maps $[g]_{\mathbb{R}^{md}}$ above must be orthogonal, i.e., belong to the orthogonal group $\mathrm{O}(md) = \mathrm{O}(\mathbb{R}^{md})$. We note that $f \mapsto f|_{\mathbb{S}}$ gives a $\mathbb{K}$-vector space isomorphism between $\mathrm{Harm}_k(\mathbb{F}^d,\mathbb{K})$ and $\mathrm{Harm}_k(\mathbb{F}^d,\mathbb{K})|_{\mathbb{S}}$, with terms **solid** and **surface** used if it is necessary to distinguish between them. The basic principles in play are:

- We mostly consider polynomials $\mathbb{H}^d \to \mathbb{C}$, since $\mathbb{H}$-valued polynomials do not commute, and there is a well developed theory of representations over $\mathbb{C}$.
- Smaller subgroups G of $\mathrm{O}(md)$ give smaller irreducible subspaces, which may lead to finer decompositions (more irreducibles).
- Enlarging the field $\mathbb{K}$ (to $\mathbb{C}$ or $\mathbb{H}$) preserves invariance of subspaces, but may not preserve irreducibility, which may lead to finer decompositions (Example 5.1).
- The irreducibles that are involved in a decomposition are of interest. The sum of all subspaces isomorphic to a given irreducible is called the *homogeneous* or *isotypic component* (for the irreducible), and it is unique. As an extreme case, all the irreducibles for the action of the trivial group are the 1-dimensional subspaces, and there is a single (uninteresting) homogeneous component.

- The reproducing kernel $K(x, y)$ for a unitarily invariant polynomial space should depend only on the inner product $\langle x, y\rangle$.
- We begin with general homogeneous polynomials for which there are natural (monomial) bases and useful inner products. We then specialise to those which are harmonic, and then, ultimately, zonal.

To develop explicit formulas, we use the Cayley-Dickson identifications $\mathbb{C}^d \cong \mathbb{R}^{2d}$ of (2.4), and $\mathbb{H}^d \cong \mathbb{C}^{2d}$ given by

$$x + iy \in \mathbb{C}^d \quad \longleftrightarrow \quad [x + iy]_{\mathbb{R}} := [x + iy]_{\mathbb{R}^{2d}} = (x, y) \in \mathbb{R}^{2d}, \tag{2.7}$$

$$z + jw \in \mathbb{H}^d \quad \longleftrightarrow \quad [z + jw]_{\mathbb{C}} := (z, w) \in \mathbb{C}^{2d}. \tag{2.8}$$

In particular, the identification (2.8) ensures that $[\cdot]_{\mathbb{C}} : \mathbb{H}^d \to \mathbb{C}^{2d}$ is $\mathbb{C}$-linear, for $\mathbb{H}^d$ as a right vector space, i.e.,

$$[(z + jw)\alpha]_{\mathbb{C}} = [z\alpha + jw\alpha]_{\mathbb{C}} = \begin{pmatrix} z\alpha \\ w\alpha \end{pmatrix} = \begin{pmatrix} z \\ w \end{pmatrix} \alpha = [(z + jw)]_{\mathbb{C}}\, \alpha, \qquad \alpha \in \mathbb{C}.$$

It is convenient to define an identification $\mathbb{H}^d \cong \mathbb{R}^{4d}$ by

$$[z + jw]_{\mathbb{R}} := [[(z + jw)]_{\mathbb{C}}]_{\mathbb{R}} = [(z, w)]_{\mathbb{R}} = (\mathrm{Re}(z), \mathrm{Re}(w), \mathrm{Im}(z), \mathrm{Im}(w)). \tag{2.9}$$

We note that

$$[z + jw]_{\mathbb{R}^{4d}} = (\mathrm{Re}(z), \mathrm{Im}(z), \mathrm{Re}(w), -\mathrm{Im}(w)].$$

We will use standard multi-index notation for monomials of degree k, e.g.,

$$z^\alpha := z_1^{\alpha_1} z_2^{\alpha_2} \cdots z_d^{\alpha_d}, \qquad |\alpha| := \alpha_1 + \cdots + \alpha_d, \qquad \alpha \in \mathbb{Z}_+^d.$$

By a dimension count, the $\binom{k+4d-1}{4d-1}$ monomials

$$m_a : \mathbb{H}^d \to \mathbb{C} : z + jw \mapsto z^{a_1} w^{a_2} \overline{z}^{a_3} \overline{w}^{a_4}, \qquad |a| = k, \quad a = (a_1, \ldots, a_4) \in \mathbb{Z}_+^{4d}, \tag{2.10}$$

are a basis for $\mathrm{Hom}_k(\mathbb{H}^d, \mathbb{C})$. We will often write $z^{a_1} w^{a_2} \overline{z}^{a_3} \overline{w}^{a_4}$ for the monomial m_a.

For $z = x + iy \in \mathbb{C}$, we define the **Wirtinger derivatives** in the usual way, i.e.,

$$\frac{\partial}{\partial z} := \frac{1}{2}\Big(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\Big), \qquad \frac{\partial}{\partial \overline{z}} := \frac{1}{2}\Big(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\Big). \tag{2.11}$$

Let Δ be the **Laplacian** operator on functions $\mathbb{H}^d \to \mathbb{C}$, which is given by

$$\frac{1}{4}\Delta = \sum_{j=1}^{d} \Big(\frac{\partial^2}{\partial \overline{z_j} \partial z_j} + \frac{\partial^2}{\partial \overline{w_j} \partial w_j} \Big). \tag{2.12}$$

By applying this, we see that the monomials

$$z^\alpha w^\beta, \qquad \overline{z}^\alpha \overline{w}^\beta, \qquad z^\alpha \overline{w}^\beta, \qquad \overline{z}^\alpha w^\beta,$$

are harmonic, i.e., in the kernel of Δ.

Let $U \in \mathbb{C}^{4d\times 4d}$ be the unitary matrix given by

$$[z+jw]_{\mathbb{R}} = \begin{pmatrix} \mathrm{Re}(z)\\ \mathrm{Re}(w)\\ \mathrm{Im}(z)\\ \mathrm{Im}(w)\end{pmatrix} = U\begin{pmatrix} z\\ w\\ \overline{z}\\ \overline{w}\end{pmatrix}, \qquad z,w\in\mathbb{C}, \qquad U=\frac{1}{2}\begin{pmatrix} 1 & 0 & 1 & 0\\ 0 & 1 & 0 & 1\\ -i & 0 & i & 0\\ 0 & -i & 0 & i\end{pmatrix}.$$

Example 2.1. Right scalar multiplication of $\mathbb{H}^d$ by $\alpha+j\beta\in\mathbb{H}$ under these identifications is given by

$$\begin{aligned}
&\mathbb{H}^d\to\mathbb{H}^d : z+jw\mapsto (z+jw)(\alpha+j\beta) = (z\alpha-\overline{w}\beta)+j(\overline{z}\beta+w\alpha),\\
&\mathbb{C}^{2d}\to\mathbb{C}^{2d} : \begin{pmatrix} z\\ w\end{pmatrix}\mapsto\begin{pmatrix} z\alpha-\overline{w}\beta\\ \overline{z}\beta+w\alpha\end{pmatrix},\\
&\mathbb{R}^{4d}\to\mathbb{R}^{4d} : [z+jw]_{\mathbb{R}}\mapsto M_{\alpha+j\beta}[z+jw]_{\mathbb{R}},
\end{aligned}$$

$$M_{\alpha+j\beta} = U\begin{pmatrix} \alpha & 0 & 0 & -\beta\\ 0 & \alpha & \beta & 0\\ 0 & -\overline{\beta} & \overline{\alpha} & 0\\ \overline{\beta} & 0 & 0 & \overline{\alpha}\end{pmatrix}\otimes IU^* = \frac{1}{2}\begin{pmatrix} \mathrm{Re}(\alpha) & -\mathrm{Re}(\beta) & -\mathrm{Im}(\alpha) & -\mathrm{Im}(\beta)\\ \mathrm{Re}(\beta) & \mathrm{Re}(\alpha) & \mathrm{Im}(\beta) & -\mathrm{Im}(\alpha)\\ \mathrm{Im}(\alpha) & -\mathrm{Im}(\beta) & \mathrm{Re}(\alpha) & \mathrm{Re}(\beta)\\ \mathrm{Im}(\beta) & \mathrm{Im}(\alpha) & -\mathrm{Re}(\beta) & \mathrm{Re}(\alpha)\end{pmatrix}\otimes I,$$

where $I=I_d$ is the $d\times d$ identity matrix. Note that this map is only $\mathbb{R}$-linear, and so there are no matrix representations for it as a map from $\mathbb{H}^d\to\mathbb{H}^d$ or $\mathbb{C}^{2d}\to\mathbb{C}^{2d}$.

Example 2.2. Consider left multiplication by a linear map $L=A+jB$, $A,B\in\mathbb{C}^{d\times d}$. By (2.1), we obtain the following matrix representations under our identifications

$$\begin{aligned}
&\mathbb{H}^d\to\mathbb{H}^d : z+jw\mapsto (A+jB)(z+jw) = Az-\overline{B}w+j(Bz+\overline{A}w),\\
&\mathbb{C}^{2d}\to\mathbb{C}^{2d} : \begin{pmatrix} z\\ w\end{pmatrix}\mapsto\begin{pmatrix} A & -\overline{B}\\ B & \overline{A}\end{pmatrix}\begin{pmatrix} z\\ w\end{pmatrix},\\
&\mathbb{R}^{4d}\to\mathbb{R}^{4d} : [z+jw]_{\mathbb{R}}\mapsto M_{A,B}[z+jw]_{\mathbb{R}},
\end{aligned}$$

$$M_{A,B} = U\begin{pmatrix} A & -\overline{B} & 0 & 0\\ B & \overline{A} & 0 & 0\\ 0 & 0 & \overline{A} & -B\\ 0 & 0 & \overline{B} & A\end{pmatrix}U^* = \frac{1}{2}\begin{pmatrix} \mathrm{Re}(A) & -\mathrm{Re}(B) & -\mathrm{Im}(A) & -\mathrm{Im}(B)\\ \mathrm{Re}(B) & \mathrm{Re}(A) & -\mathrm{Im}(B) & \mathrm{Im}(A)\\ \mathrm{Im}(A) & \mathrm{Im}(B) & \mathrm{Re}(A) & -\mathrm{Re}(B)\\ \mathrm{Im}(B) & -\mathrm{Im}(A) & \mathrm{Re}(B) & \mathrm{Re}(A)\end{pmatrix}.$$

Matrix multiplication on the left (which includes left scalar multiplication) commutes with right scalar multiplication, by the associative law

$$(Lv)\alpha = L(v\alpha), \qquad v\in\mathbb{H}^d,\ \alpha\in\mathbb{H}. \tag{2.13}$$

Conversely, those matrices in $\mathbb{R}^{4d}\times\mathbb{R}^{4d}$ which commute with all $M_{\alpha+j\beta}$ (equivalently M_1, M_i, M_j and M_k) correspond to the matrices $L\in\mathbb{H}^{d\times d}$, and are said to be **symplectic**.

The **compact symplectic group** $\mathrm{Sp}(d)$, **quaternionic unitary group** $U(\mathbb{H}^d)$ or **hyperunitary group** (see [14] §1.2.8) is the group of unitary matrices in $\mathbb{H}^{d\times d}$ for the inner product (2.2), or, equivalently, the symplectic matrices in $\mathbb{R}^{4d\times 4d}$ which are orthogonal. These may also be viewed as the unitary matrices of the form

$$\mathbb{C}^{2d}\to\mathbb{C}^{2d} : \begin{pmatrix} z\\ w\end{pmatrix}\mapsto\begin{pmatrix} A & -\overline{B}\\ B & \overline{A}\end{pmatrix}\begin{pmatrix} z\\ w\end{pmatrix}, \qquad A^*A+B^*B=I, \quad A^TB-B^TA=0.$$

In particular, $\mathrm{Sp}(1)=U(\mathbb{H})$ is the group of unit quaternions or, the special unitary group

$$SU(2) = \left\{ \begin{pmatrix} \alpha & -\overline{\beta} \\ \beta & \overline{\alpha} \end{pmatrix} : |\alpha|^2 + |\beta|^2 = 1, \alpha, \beta \in \mathbb{C} \right\},$$

which therefore have the same irreducible representations (see [12] §5.4).

3. The operators R and L

A subspace V of $\mathrm{Harm}_k(\mathbb{H}^d, \mathbb{C})$ is **invariant** under the right multiplication by

$$\mathbb{H}^* := \mathbb{H} \setminus \{0\},$$

equivalently $U(\mathbb{H})$, if $f(z + jw) \in V$ implies $f\big((z + jw)(\alpha + j\beta)\big) \in V$, and similarly for left multiplication. The following elementary example shows how we came to the operators R and L.

Example 3.1. Suppose that $V \subset \mathrm{Harm}_3(\mathbb{H}^1, \mathbb{C})$ is invariant under right multiplication by $\mathbb{H}^*$, and $f\big((z + jw)\big) = z^2 w \in V$, then

$$\begin{aligned} f\big((z + jw)(\alpha + j\beta)\big) &= (z\alpha - \overline{w}\beta)^2(\overline{z}\beta + w\alpha) \\ &= \alpha^3 z^2 w + \alpha^2\beta(z^2\overline{z} - 2zw\overline{w}) + \alpha\beta^2(w\overline{w}^2 - 2z\overline{zw}) + \beta^3\overline{zw}^2 \in V. \end{aligned}$$

By taking different choices for α and β, it is not hard to see that the "coefficients" of the monomials $\alpha^3, \alpha^2\beta, \alpha\beta^2, \beta^3$ above are in V, i.e.,

$$z^2 w, z^2\overline{z} - 2zw\overline{w}, w\overline{w}^2 - 2z\overline{zw}, \overline{zw}^2 \in V.$$

Similarly, any partial derivative with respect to α or β will be a linear combination of the above polynomials, and hence in V.

This example naturally generalises as follows.

Lemma 3.2. *Let $f \in \mathrm{Hom}_k(\mathbb{H}^d, \mathbb{C})$ be given by $f = f(z + jw) = F(z, w, \overline{z}, \overline{w})$, and V_f be the subspace invariant under right multiplication by $\mathbb{H}^*$ generated by f. Then V_f contains*

$$f\left((z + jw)(\alpha + j\beta)\right) = F(z\alpha - \overline{w}\beta, \overline{z}\beta + w\alpha, \overline{z\alpha} - w\overline{\beta}, z\overline{\beta} + \overline{w\alpha}), \quad \alpha, \beta \in \mathbb{C},$$

and all its partial derivatives in the variables $\alpha, \overline{\alpha}, \beta, \overline{\beta}$, including

$$R_a f := \frac{\partial^{a_1+a_2+a_3+a_4}}{\partial\alpha^{a_1}\partial\beta^{a_2}\partial\overline{\alpha}^{a_3}\partial\overline{\beta}^{a_4}} f\left((z + jw)(\alpha + j\beta)\right)\Big|_{\alpha=1,\beta=0}. \tag{3.1}$$

Moreover, if f is harmonic, then so are all the polynomials in V_f.

Proof. Clearly, V_f is the subspace of $\mathrm{Hom}_k(\mathbb{H}^d, \mathbb{C})$ given by

$$V_f = \mathrm{span}_{\mathbb{C}}\{f\left((z + jw)q\right) : q = \alpha + j\beta \in \mathbb{H}^*\},$$

and hence for $q = \alpha + j\beta$ nonzero, we have

$$f\left((z + jw)(\alpha + j\beta)\right) = |q|^k f((z + jw)(q/|q|)) \in V_f.$$

Since V_f is a finite-dimensional vector space (and hence is closed for any norm), it follows that the first order partials in $\alpha, \overline{\alpha}, \beta, \overline{\beta}$, which are limits of Newton quotients in V_f, are in V_f, and therefore so are all the partial derivatives.

A calculation shows that if $f : \mathbb{R}^n \to \mathbb{R}$ is harmonic, then so is $f \circ U$ for $U : \mathbb{R}^n \to \mathbb{R}^n$ orthogonal. Since scalar multiplication of $\mathbb{H}^d$ by a unit quaternion (left or right) is an orthogonal map $\mathbb{R}^{4d} \to \mathbb{R}^{4d}$, it follows that if $f(z + jw)$ is harmonic, then so is $f((z + jw)q)$, $q \in \mathbb{H}^*$, and hence every polynomial in V_f. $\square$

In other words, if a subspace $V \subset \mathrm{Hom}_k(\mathbb{H}^d, \mathbb{C})$ is invariant under right multiplication by $\mathbb{H}^*$, then it is invariant under the action of the operators R_a of (3.1). Since the partial derivatives in (3.1) for $|a| = a_1 + a_2 + a_3 + a_4 = k$ do not depend on $\alpha, \beta, \overline{\alpha}, \overline{\beta}$, they can be "evaluated" at $\alpha = 0, \beta = 0$, to obtain the Taylor formula

$$f((z + jw)(\alpha + j\beta)) = \sum_{|a|=k} R_a(f) \frac{\alpha^{a_1} \beta^{a_2} \overline{\alpha}^{a_3} \overline{\beta}^{a_4}}{a_1! a_2! a_3! a_4!}, \quad f \in \mathrm{Hom}_k(\mathbb{H}^d, \mathbb{C}). \tag{3.2}$$

We therefore have the following converse result.

Proposition 3.3. *A subspace $V \subset \mathrm{Hom}_k(\mathbb{H}^d, \mathbb{C})$ is invariant under right multiplication by $\mathbb{H}^*$ if and only if it is invariant under the operators R_a, $|a| = k$.*

Proof. As already observed, the forward implication follows from Lemma 3.2.

Conversely, suppose that V is invariant under right multiplication by $\mathbb{H}^*$, and $f \in V$. Since the monomials in $\alpha, \beta, \overline{\alpha}, \overline{\beta}$ in the Taylor formula (3.2) are linearly independent, it follows that

$$V_f = \mathrm{span}_{\mathbb{C}}\{R_a f : |a| = k\} \subset V,$$

and so V is invariant under right multiplication by the operators R_a, $|a| = k$. $\square$

There is an obvious parallel development for the left multiplication by $\mathbb{H}^*$ where the role of R_a is played by L_a, where

$$f((\alpha + j\beta)(z + jw)) = F(\alpha z - \overline{\beta} w, \overline{\alpha} w + \beta z, \overline{\alpha z} - \beta \overline{w}, \alpha \overline{w} + \overline{\beta z}),$$
$$L_a f := \frac{\partial^{a_1+a_2+a_3+a_4}}{\partial \alpha^{a_1} \partial \beta^{a_2} \partial \overline{\alpha}^{a_3} \partial \overline{\beta}^{a_4}} f((\alpha + j\beta)(z + jw)) \Big|_{\alpha=1, \beta=0}. \tag{3.3}$$

Example 3.4. For the Example 3.1, i.e., $f = z^2 w$, the nonzero terms in (3.2) are

$$R_{3,0,0,0} f = 6z^2 w, \quad R_{2,1,0,0} f = 2z^2 \overline{z} - 4zw\overline{w}, \quad R_{1,2,0,0} f = 2w\overline{w}^2 - 4z\overline{zw}, \quad R_{0,3,0,0} f = 6\overline{zw}^2.$$

The nonzero polynomials $R_a f$ are not a basis for V_f in general, e.g., for $f = z\overline{w}$ one has

$$R_{1,0,1,0} f = z\overline{w}, \quad R_{0,1,0,1} f = -z\overline{w}, \quad R_{1,0,0,1} f = z^2, \quad R_{0,1,1,0} f = -\overline{w}^2.$$

There are too many operators R_a for a practicable theory, and so we seek a smaller subset of "generators". By the same argument of Lemma 3.2, from

$$\begin{aligned} &f((z + jw)(1 + (\alpha + j\beta)t)) \\ &\quad = F\left(z(1 + \alpha t) - \overline{w}\beta t, \overline{z}\beta t + w(1 + \alpha t), \overline{z}(1 + \overline{\alpha} t) - w\overline{\beta} t, z\overline{\beta} t + \overline{w}(1 + \overline{\alpha})\right), \quad t \in \mathbb{R}, \end{aligned}$$

we can define first order linear differential operators $R_\alpha, R_\beta, R_{\overline{\alpha}}, R_{\overline{\beta}}$ by

$$\frac{d}{dt} f\big((z+jw)(1+(\alpha+j\beta)t)\big)\Big|_{t=0} = \alpha R_\alpha f + \beta R_\beta f + \overline{\alpha} R_{\overline{\alpha}} f + \overline{\beta} R_{\overline{\beta}} f. \tag{3.4}$$

For these, $R_\alpha f$, $R_\beta f$, $R_{\overline{\alpha}} f$ and $R_{\overline{\beta}} f$ belong to V_f (the subspace generated by f which is invariant under right multiplication by $\mathbb{H}^*$). The subscripts of R are symbolic, i.e., the operators do not depend on $\alpha + j\beta$, e.g.,

$$R_\beta = -\overline{w}\frac{\partial}{\partial z} + \overline{z}\frac{\partial}{\partial w} \quad (d=1), \qquad R_\beta = \sum_{j=1}^d \Big(-\overline{w_j}\frac{\partial}{\partial z_j} + \overline{z_j}\frac{\partial}{\partial w_j}\Big). \tag{3.5}$$

It is the case that

$$R_\alpha f = R_{1,0,0,0} f = \frac{\partial}{\partial \alpha} f\big((z+jw)(\alpha+j\beta)\big)\Big|_{\alpha=1,\beta=0}, \quad R_\beta f = R_{0,1,0,0} f,$$

etc. The analogous operators $L_\alpha, L_\beta, L_{\overline{\alpha}}, L_{\overline{\beta}}$ for left multiplication by $\mathbb{H}^*$ are defined by

$$\frac{d}{dt} f\big((z+jw)(1+(\alpha+j\beta)t)\big)\Big|_{t=0} = \alpha L_\alpha f + \beta L_\beta f + \overline{\alpha} L_{\overline{\alpha}} f + \overline{\beta} L_{\overline{\beta}} f. \tag{3.6}$$

All of these first order operators T satisfy the product rule

$$T(fg) = T(f)g + fT(g). \tag{3.7}$$

For readability, we will often present them and do calculations in the $d = 1$ case, with the general case following by replacing z by z_j, etc, and summing over j, as in (3.5). The operators are (for $d = 1$)

$$R_\beta = -\overline{w}\frac{\partial}{\partial z} + \overline{z}\frac{\partial}{\partial w}, \qquad R_{\overline{\beta}} = -w\frac{\partial}{\partial \overline{z}} + z\frac{\partial}{\partial \overline{w}}, \tag{3.8}$$

$$R_\alpha = z\frac{\partial}{\partial z} + w\frac{\partial}{\partial w}, \qquad R_{\overline{\alpha}} = \overline{z}\frac{\partial}{\partial \overline{z}} + \overline{w}\frac{\partial}{\partial \overline{w}}, \tag{3.9}$$

$$L_\beta = z\frac{\partial}{\partial w} - \overline{w}\frac{\partial}{\partial \overline{z}}, \qquad L_{\overline{\beta}} = \overline{z}\frac{\partial}{\partial \overline{w}} - w\frac{\partial}{\partial z}, \tag{3.10}$$

$$L_\alpha = z\frac{\partial}{\partial z} + \overline{w}\frac{\partial}{\partial \overline{w}}, \qquad L_{\overline{\alpha}} = w\frac{\partial}{\partial w} + \overline{z}\frac{\partial}{\partial \overline{z}}. \tag{3.11}$$

Of particular interest, are the operators

$$R := -R_\beta, \qquad R^* := R_{\overline{\beta}}, \qquad L := -L_{\overline{\beta}}, \qquad L^* := L_\beta, \tag{3.12}$$

which in the 1-dimensional case have the form

$$R = \overline{w}\frac{\partial}{\partial z} - \overline{z}\frac{\partial}{\partial w}, \qquad R^* = -w\frac{\partial}{\partial \overline{z}} + z\frac{\partial}{\partial \overline{w}}, \tag{3.13}$$

$$L = w\frac{\partial}{\partial z} - \overline{z}\frac{\partial}{\partial \overline{w}}, \qquad L^* = z\frac{\partial}{\partial w} - \overline{w}\frac{\partial}{\partial \overline{z}}. \tag{3.14}$$

For a general d, we have

$$R = \sum_{j=1}^d \Big(\overline{w_j}\frac{\partial}{\partial z_j} - \overline{z_j}\frac{\partial}{\partial w_j}\Big), \qquad R^* = \sum_{j=1}^d \Big(-w_j\frac{\partial}{\partial \overline{z_j}} + z_j\frac{\partial}{\partial \overline{w_j}}\Big). \tag{3.15}$$

The notation R^* and L^* is used, as we will see (Lemma 4.1) that they are the adjoints of R and L, respectively, for two natural inner products. The operators R and R^* (but not L and L^*) appear in the work of [4], [5] as $\varepsilon = R^*$ and $\varepsilon^\dagger = R$. Operators of this type (for $d = 1$) also appear in the construction of irreducible representations of $\mathrm{SU}(2) \subset \mathbb{H}^*$ on the homogeneous polynomials in z and w of degree k given in [12].

Elementary calculations [20] give the following:

Proposition 3.5. *We have the commutativity relations*

$$R_\beta R_{\overline{\beta}} - R_{\overline{\beta}} R_\beta = R_\alpha - R_{\overline{\alpha}}, \qquad R_\alpha R_{\overline{\alpha}} = R_{\overline{\alpha}} R_\alpha, \tag{3.16}$$

$$R_\beta R_\alpha - R_\alpha R_\beta = R_\beta, \qquad R_\alpha R_{\overline{\beta}} - R_{\overline{\beta}} R_\alpha = R_{\overline{\beta}}, \tag{3.17}$$

$$R_{\overline{\alpha}} R_\beta - R_\beta R_{\overline{\alpha}} = R_\beta, \qquad R_{\overline{\beta}} R_{\overline{\alpha}} - R_{\overline{\alpha}} R_{\overline{\beta}} = R_{\overline{\beta}}. \tag{3.18}$$

Furthermore, on $\mathrm{Hom}_k(\mathbb{H}^d, \mathbb{C})$ *we have*

$$R_\alpha + R_{\overline{\alpha}} = kI, \tag{3.19}$$

and hence

$$R_\alpha = \frac{1}{2}(R^*R - RR^* + kI), \qquad R_{\overline{\alpha}} = \frac{1}{2}(RR^* - R^*R + kI). \tag{3.20}$$

It was hoped use these formulas to show that on $\mathrm{Hom}_k(\mathbb{H}^d, \mathbb{C})$ each R_a, $|a| = k$, in (3.2) could be written as a polynomial in the noncommuting variables R and R^*, so that invariance under right multiplication by $\mathbb{H}^*$ is equivalent to invariance under R and R^*. However, a completely general formula for $R_a f$ could not be obtained. A direct, but more abstract, route to this conclusion comes from Lie theory, which we now discuss.

The group $G = Sp(1)$ is a compact simply connected real Lie group, and its Lie algebra $\mathfrak{g}$ is the quaternions with $q + \overline{q} = 0$, i.e., zero real part, which are often called "vectors" because of the identification of $q = v = ai + bj + ck$ with $(a, b, c) \in \mathbb{R}^3$, where the Lie bracket becomes the cross product of vectors. Moreover, G is matrix Lie group, i.e., is a closed subgroup of $GL_2(\mathbb{C})$, and so the following general result applies.

Proposition 3.6. *([14]) Let G be a matrix Lie group with Lie algebra $\mathfrak{g}$, and Π be a finite-dimensional real or complex representation of G acting on a vector space V. Then there is a unique (Lie algebra) representation π of $\mathfrak{g}$ acting on V such that*

$$\Pi(e^X) = e^{\pi(X)}, \qquad \forall X \in \mathfrak{g},$$

which is given by

$$\pi(X) = \frac{d}{dt}\Pi(e^{tX})\Big|_{t=0}. \tag{3.21}$$

When G is connected, there is a 1–1 correspondence between irreducibles.

Proposition 3.7. *([14]) Let G be a connected matrix Lie group with Lie algebra $\mathfrak{g}$. Then*

1. *A representation Π of G is irreducible if and only if the associated representation π of $\mathfrak{g}$ is irreducible.*
2. *Two representations Π_1 and Π_2 of G are isomorphic if and only if the associated representations π_1 and π_2 of $\mathfrak{g}$ are isomorphic.*

Let $G = \mathrm{Sp}(1)$, $V = \mathrm{Hom}_k(\mathbb{H}^d, \mathbb{C})$ and $\Pi : G \to GL(V)$ be the representation of G induced by right multiplication of $\mathbb{H}^d$ by $\mathbb{H}^*$, i.e.,

$$(\alpha + j\beta) \cdot f(z + jw) := f\big((z + jw)(\alpha + j\beta)\big).$$

This is a right action (so, technically speaking, G should be given the multiplication of the opposite groups so that Π is a homomorphism, but this just complicates the formulas and their derivation), which for $f \in V$ given by $f = f(z + jw) = F(z, w, \overline{z}, \overline{w})$ is

$$\Pi(\alpha + j\beta)f(z + jw) = F(z\alpha - \overline{w}\beta, \overline{z}\beta + w\alpha, \overline{z\alpha} - w\overline{\beta}, z\overline{\beta} + \overline{w\alpha}), \quad \alpha, \beta \in \mathbb{C}.$$

For a "vector" $X = v = ai + bj + ck \in \mathfrak{g}$, we have (see [19]) that

$$e^{tv} = e^{t(ai+bj+ck)} = \cos(t|v|) + \frac{ai + bj + ck}{|v|}\sin(t|v|) = \alpha_t + j\beta_t, \quad \alpha_t, \beta_t \in \mathbb{C}, \tag{3.22}$$

where

$$\alpha_t := \cos(t|v|) + \frac{ai}{|v|}\sin(t|v|), \qquad \beta_t := \frac{b - ci}{|v|}\sin(t|v|).$$

Since

$$\alpha_0 = 1, \quad \beta_0 = 0, \qquad \frac{d}{dt}\alpha_t\Big|_{t=0} = ai, \quad \frac{d}{dt}\beta_t\Big|_{t=0} = b - ci,$$

we calculate (3.21) for $d = 1$

$$\begin{aligned}
\pi(v)f(z + jw) &= \frac{d}{dt}F(z\alpha_t - \overline{w}\beta_t, \overline{z}\beta_t + w\alpha_t, \overline{z\alpha_t} - w\overline{\beta_t}, z\overline{\beta_t} + \overline{w\alpha_t})\Big|_{t=0} \\
&= (zai - \overline{w}(b - ci))\frac{\partial f}{\partial z} + (\overline{z}(b - ci) + wai)\frac{\partial f}{\partial w} \\
&\quad + (\overline{z}\overline{ai} - w\overline{(b - ci)})\frac{\partial f}{\partial \overline{z}} + (z\overline{(b - ci)} + \overline{w}\overline{ai})\frac{\partial f}{\partial \overline{w}} \\
&= ai\Big(z\frac{\partial}{\partial z} + w\frac{\partial}{\partial w} - \overline{z}\frac{\partial}{\partial \overline{z}} - \overline{w}\frac{\partial}{\partial \overline{w}}\Big) + b\Big(-\overline{w}\frac{\partial}{\partial z} + \overline{z}\frac{\partial}{\partial w} - w\frac{\partial}{\partial \overline{z}} + z\frac{\partial}{\partial \overline{w}}\Big)f \\
&\quad + ci\Big(\overline{w}\frac{\partial}{\partial z} - \overline{z}\frac{\partial}{\partial w} - w\frac{\partial}{\partial \overline{z}} + z\frac{\partial}{\partial \overline{w}}\Big)f,
\end{aligned}$$

i.e.,

$$\pi(ai + bj + ck)f(z + jw) = ai(R_\alpha - R_{\overline{\alpha}})f + b(R_\beta + R_{\overline{\beta}})f + ci(-R_\beta + R_{\overline{\beta}})f, \tag{3.23}$$

which holds for all d, by the same argument. From this we obtain the desired result.

Lemma 3.8. *(Lie correspondence) A subspace of* $\mathrm{Hom}_k(\mathbb{H}^d, \mathbb{C})$ *is invariant under the action of right multiplication by* $\mathbb{H}^*$*, equivalently by* $\mathrm{Sp}(1)$*, if and only if it is invariant under the action of* R *and* R^**. Moreover, the irreducibles for these actions are the same.*

Proof. It follows from Proposition 3.7, that a subspace of $\mathrm{Hom}_k(\mathbb{H}^d, \mathbb{C})$ is invariant under right multiplication by $\mathrm{Sp}(1)$ (and hence $\mathbb{H}^*$) if and only if it is invariant under the action of π given by (3.23), i.e., it is invariant under the span of the operators

$$R_\alpha - R_{\overline{\alpha}}, \quad R_\beta + R_{\overline{\beta}}, \quad -R_\beta + R_{\overline{\beta}}.$$

Since

$$R = -R_\beta = -\frac{1}{2}\Big((R_\beta + R_{\overline{\beta}}) - (-R_\beta + R_{\overline{\beta}})\Big), \quad R^* = R_{\overline{\beta}} = \frac{1}{2}\Big((R_\beta + R_{\overline{\beta}}) + (-R_\beta + R_{\overline{\beta}})\Big),$$

and, by (3.20), we have

$$R_\alpha - R_{\overline{\alpha}} = R^*R - RR^*,$$

this is equivalent to invariance under the action of R and R^*. $\square$

The analogous statement holds for left multiplication and the action of L and L^*.

The Lie representation $\pi : \mathfrak{g} \to GL(V)$ preserves the Lie bracket $[\cdot,\cdot]$, which is the commutator

$$[A, B] = AB - BA,$$

in each case. In particular, since $[j, k] = jk - kj = 2i$, we have

$$2\pi(i) = [\pi(j), \pi(k)] \quad \Longleftrightarrow \quad 2i(R_\alpha - R_{\overline{\alpha}}) = [R_\beta + R_{\overline{\beta}}, i(-R_\beta + R_{\overline{\beta}})],$$

which gives (3.16), i.e.,

$$R_\alpha - R_{\overline{\alpha}} = \frac{1}{2}[R_\beta + R_{\overline{\beta}}, -R_\beta + R_{\overline{\beta}}] = R_\beta R_{\overline{\beta}} - R_{\overline{\beta}} R_\beta = [R_\beta, R_{\overline{\beta}}].$$

We now investigate the formula for the action of G on V in terms of R and R^*. For $\alpha + j\beta \in \mathrm{Sp}(1)$, (3.22) gives

$$e^v = \alpha + j\beta, \qquad v = ai + bj + ck,$$

where

$$\alpha = \cos(|v|) + \frac{ai}{|v|}\sin(|v|), \qquad \beta := \frac{b - ci}{|v|}\sin(|v|),$$

i.e.,

$$\cos|v| = \frac{\alpha + \overline{\alpha}}{2} \quad \Longrightarrow \quad |v| = \cos^{-1}\frac{\alpha + \overline{\alpha}}{2}, \tag{3.24}$$

$$\frac{a}{|v|}\sin|v| = \frac{\alpha - \overline{\alpha}}{2i}, \qquad \frac{b}{|v|}\sin|v| = \frac{\beta + \overline{\beta}}{2}, \qquad -\frac{c}{|v|}\sin|v| = \frac{\beta - \overline{\beta}}{2i}. \tag{3.25}$$

If $\mathrm{Re}(\alpha) = \frac{\alpha + \overline{\alpha}}{2} \neq \pm 1$, then $\sin|v| \neq 0$, and (3.23) and (3.25) give

$$\pi(v)f(z + jw) = \frac{|v|}{\sin|v|}\Big(\frac{\alpha - \overline{\alpha}}{2i} i(R_\alpha - R_{\overline{\alpha}})f + \frac{\beta + \overline{\beta}}{2}(R_\beta + R_{\overline{\beta}})f - \frac{\beta - \overline{\beta}}{2i} i(-R_\beta + R_{\overline{\beta}})f\Big),$$

where $|v|$ is the function of α and $\overline{\alpha}$ given by (3.24). Thus, we have

$$\pi(v) = \frac{|v|}{2\sin|v|}\Big((\alpha - \overline{\alpha})(R^*R - RR^*) + (\beta + \overline{\beta})(-R + R^*) - (\beta - \overline{\beta})(R + R^*)\Big), \tag{3.26}$$

and

$$f\big((z+jw)(\alpha+j\beta)\big) = e^{\pi(v)} f. \tag{3.27}$$

4. Inner products on the quaternionic sphere

There are two natural (unitarily invariant) inner products defined on polynomials from $\mathbb{H}^d \to \mathbb{C}$ that we consider. Let

$$\mathbb{S} = \mathbb{S}(\mathbb{F}^d) := \{x \in \mathbb{F}^d : \|x\| = 1\} = \{x \in \mathbb{R}^{md} : \|x\| = 1\}$$

be the unit sphere in $\mathbb{F}^d$, and σ be the surface area measure on $\mathbb{S}$, normalised so that $\sigma(\mathbb{S}) = 1$. We note that surface area measure is invariant under unitary maps on $\mathbb{F}^d$, i.e., for U unitary

$$\int_{\mathbb{S}(\mathbb{F}^d)} f(Ux)\, d\sigma(x) = \int_{\mathbb{S}(\mathbb{F}^d)} f(x)\, d\sigma(x), \qquad \forall f.$$

The first inner product we consider is defined on complex-valued functions restricted to the **quaternionic sphere** $\mathbb{S} = \mathbb{S}(\mathbb{H}^d)$ by

$$\langle f, g\rangle = \langle f, g\rangle_{\mathbb{S}} := \int_{\mathbb{S}(\mathbb{H}^d)} \overline{f(x)} g(x)\, d\sigma(x). \tag{4.1}$$

This can be calculated from the well known integrals of the monomials in $z, w, \overline{z}, \overline{w} \in \mathbb{C}^d$ (polynomials in $2d$ complex variables)

$$\int_{\mathbb{S}(\mathbb{H}^d)} z^{\alpha_1} w^{\beta_1} \overline{z}^{\alpha_2} \overline{w}^{\beta_2}\, d\sigma = \int_{\mathbb{S}(\mathbb{C}^{2d})} z^{\alpha_1} w^{\beta_1} \overline{z}^{\alpha_2} \overline{w}^{\beta_2}\, d\sigma(z,w),$$

which are zero for $(\alpha_1, \beta_1) \neq (\alpha_2, \beta_2)$, and otherwise

$$\int_{\mathbb{S}(\mathbb{H}^d)} z^{\alpha_1} w^{\beta_1} \overline{z}^{\alpha_2} \overline{w}^{\beta_2}\, d\sigma = \frac{(2d-1)!\alpha_1!\beta_1!}{(2d-1+|\alpha_1|+|\beta_1|)!} = \frac{\alpha_1!\beta_1!}{(2d)_{|\alpha_1|+|\beta_1|}}, \qquad (\alpha_1,\beta_1) = (\alpha_2,\beta_2). \tag{4.2}$$

Here $(x)_n := x(x+1)\cdots(x+n-1)$ is the Pochhammer symbol.

For a polynomial $f = \sum_\alpha f_\alpha z^{\alpha_1} w^{\alpha_2} \overline{z}^{\alpha_3} \overline{w}^{\alpha_4}$ mapping $\mathbb{H}^d \to \mathbb{C}$, let $\tilde{f}$ be the polynomial obtained by replacing the coefficient $f_\alpha \in \mathbb{C}$ by its conjugate $\overline{f_\alpha}$, and $f(\partial)$ be the differential operator obtained replacing z by $\frac{\partial}{\partial z}$, etc, i.e.,

$$\tilde{f} = \sum_\alpha \overline{f_\alpha} z^{\alpha_1} w^{\alpha_2} \overline{z}^{\alpha_3} \overline{w}^{\alpha_4}, \qquad f(\partial) = \sum_\alpha f_\alpha \frac{\partial^{\alpha_1+a_2+\alpha_3+\alpha_4} f}{\partial z^{\alpha_1} \partial w^{\alpha_2} \partial \overline{z}^{\alpha_3} \partial \overline{w}^{\alpha_4}}.$$

The second inner product is given by

$$\langle f, g\rangle_\partial := \tilde{f}(\partial) g(0) = \sum_\alpha \alpha! \overline{f_\alpha} g_\alpha. \tag{4.3}$$

The inner products (4.1) and (4.3) are both prominent in the theory of spherical harmonics. The first is natural for Fourier expansions on the sphere, and the second, which is variously known as the **apolar** [25],

Bombieri [29] or **Fischer inner product** [4], is also widely used. Not withstanding the fact that they are defined on different spaces, these inner products are different, since the monomials are orthogonal in the second, but not in the first in general, e.g.,

$$\langle z_1\overline{z_1}, w_1\overline{w_1}\rangle = \int_{\mathbb{S}(\mathbb{H}^d)} |z_1 w_1|^2 \, d\sigma = \frac{1}{2d(2d+1)} \neq 0, \qquad \langle z_1\overline{z_1}, w_1\overline{w_1}\rangle_\partial = 0.$$

Nevertheless, these inner product are scalar multiples of each other in the following sense

$$\langle f, g\rangle_\partial = (2d)_k \langle f, g\rangle, \qquad f \in \mathrm{Harm}_k(\mathbb{H}^d, \mathbb{C}),\ g \in \mathrm{Hom}_k(\mathbb{H}^d, \mathbb{C}),$$

which follows from [10] (Theorem 1.1.8) as presented in [9] (Lemma 2).

The homogeneous polynomials of different degrees are orthogonal to each other for both inner products, giving the orthogonal direct sums

$$\bigoplus_{k\ge 0} \mathrm{Hom}_k(\mathbb{H}^d, \mathbb{C})\Big|_{\mathbb{S}(\mathbb{H}^d)}, \qquad \bigoplus_{k\ge 0} \mathrm{Hom}_k(\mathbb{H}^d, \mathbb{C}),$$

respectively. For simplicity, we will primarily consider the further decomposition of $\mathrm{Harm}_k(\mathbb{H}^d, \mathbb{C})$, with it being understood that this leads to a corresponding refinement of the direct sums

$$\bigoplus_{k\ge 0} \mathrm{Hom}_k(\mathbb{H}^d, \mathbb{C}) = \bigoplus_{k\ge 0} \bigoplus_{0\le j\le \frac{k}{2}} \|\cdot\|^{2j}\, \mathrm{Harm}_{k-2j}(\mathbb{H}^d, \mathbb{C}), \tag{4.4}$$

$$\mathrm{Hom}_k(\mathbb{H}^d, \mathbb{C})\big|_{\mathbb{S}} = \bigoplus_{0\le j\le \frac{k}{2}} \mathrm{Harm}_{k-2j}(\mathbb{H}^d, \mathbb{C}), \tag{4.5}$$

of the polynomials $\mathbb{H}^d \to \mathbb{C}$ into irreducibles for the action of a subgroup of $\mathrm{O}(4d)$. The direct sum (4.4) is sometimes referred to as the **Fischer decomposition** [4].

The adjoints of R and L are the same for both of these inner products.

Lemma 4.1. *The operators R^* and L^* are the adjoints of R and L with respect to both the inner products (4.1) and (4.3) defined on $\mathrm{Hom}_k(\mathbb{H}^d, \mathbb{C})$.*

This result for R was given in [4] Lemma 5 for the inner product (4.3).

The adjoint can also be calculated using the following property.

Example 4.2. An elementary calculation shows the identities

$$\overline{Rf} = -R^*(\overline{f}), \qquad \overline{Lf} = -L^*(\overline{f}), \tag{4.6}$$

and so, on subspaces V, we have

$$\overline{R^\alpha V} = (R^*)^\alpha \overline{V}, \qquad \overline{\ker R^*|_V} = \ker R|_{\overline{V}}. \tag{4.7}$$

The next result follows from the fact that scalar multiplication by $\mathbb{H}^*$ is in $\mathrm{O}(\mathbb{R}^{4d})$, and hence maps harmonic polynomials to harmonic polynomials.

Lemma 4.3. *The operators R, R^*, L and L^* commute with the Laplacian Δ, and so map harmonic functions to harmonic functions.*

It follows from (2.13) that the action of R and R^* commutes with that of $U \in U(\mathbb{H}^d)$. In this regard, recall from (2.6) and (3.1) that

$$(U \cdot f)(z + jw) = f(U^{-1}(z + jw)).$$

Lemma 4.4. *The operators R and R^* commute with the action of $U(\mathbb{H}^d)$.*

Proof. We will show, more generally, that the operators R_a of (3.1) commute with the action of $U(\mathbb{H}^d)$. Let $U \in U(\mathbb{H}^d)$. Then for $f = f(z + jw)$, we have

$$\begin{aligned}(U \cdot R_a f) &= \frac{\partial^{a_1+a_2+a_3+a_4}}{\partial\alpha^{a_1}\partial\beta^{a_2}\partial\overline{\alpha}^{a_3}\partial\overline{\beta}^{a_4}} f\left(U^{-1}(z + jw)(\alpha + j\beta)\right)\Big|_{\alpha=1,\beta=0} \\ &= \frac{\partial^{a_1+a_2+a_3+a_4}}{\partial\alpha^{a_1}\partial\beta^{a_2}\partial\overline{\alpha}^{a_3}\partial\overline{\beta}^{a_4}} (U \cdot f)\left((z + jw)(\alpha + j\beta)\right)\Big|_{\alpha=1,\beta=0} \\ &= R_a(U \cdot f).\end{aligned}$$

i.e., R_a commutes with the action of $U(\mathbb{H}^d)$. □

We note that, by the same reasoning, the operators L and L^* do not commute with the (left) action of $U(\mathbb{H}^d)$.

5. The action of R and L on polynomials

Using (3.13) to apply R to a univariate monomial $f = z^{a_1}w^{a_2}\overline{z}^{a_3}\overline{w}^{a_4}$ gives

$$\begin{aligned}Rf &= \overline{w}\frac{\partial}{\partial z}(z^{a_1}w^{a_2}\overline{z}^{a_3}\overline{w}^{a_4}) - \overline{z}\frac{\partial}{\partial w}(z^{a_1}w^{a_2}\overline{z}^{a_3}\overline{w}^{a_4}) \\ &= a_1 z^{a_1-1}w^{a_2}\overline{z}^{a_3}\overline{w}^{a_4+1} - a_2 z^{a_1}w^{a_2-1}\overline{z}^{a_3+1}\overline{w}^{a_4},\end{aligned}$$

which is a sum of monomials in which the degree in z and w has decreased by 1, whilst the degree in $\overline{z}$ and $\overline{w}$ has increased by 1. This type of phenomenon occurs for all of the operators R, R^*, L, L^* (in every dimension), and we now make definitions which allow us to account for these changes in degrees. With standard multi-index notation, we have

$$\begin{aligned}\operatorname{Hom}_H(p, q) &:= \operatorname{span}\{z^{\alpha_1}w^{\alpha_2}\overline{z}^{\alpha_3}\overline{w}^{\alpha_4} : |\alpha_1| + |\alpha_2| = p, |\alpha_3| + |\alpha_4| = q\}, \\ \operatorname{Hom}_K(p, q) &:= \operatorname{span}\{z^{\alpha_1}w^{\alpha_2}\overline{z}^{\alpha_3}\overline{w}^{\alpha_4} : |\alpha_1| + |\alpha_4| = p, |\alpha_2| + |\alpha_3| = q\}, \\ \operatorname{Hom}_k^{(a,b)}(\mathbb{H}^d) &:= \operatorname{Hom}_K(k - a, a) \cap \operatorname{Hom}_H(k - b, b) \\ &= \operatorname{span}\{z^{\alpha_1}w^{\alpha_2}\overline{z}^{\alpha_3}\overline{w}^{\alpha_4} : |\alpha_1| + |\alpha_4| = k - a, |\alpha_2| + |\alpha_3| = a, \\ &\qquad |\alpha_1| + |\alpha_2| = k - b, |\alpha_3| + |\alpha_4| = b\}.\end{aligned}$$

We observe that

$$\overline{\operatorname{Hom}_H(p, q)} = \operatorname{Hom}_H(q, p), \qquad \overline{\operatorname{Hom}_K(p, q)} = \operatorname{Hom}_K(q, p).$$
$$\Delta \operatorname{Hom}_H(a, b) = \operatorname{Hom}_H(a - 1, b - 1), \qquad \Delta \operatorname{Hom}_K(a, b) = \operatorname{Hom}_K(a - 1, b - 1).$$

The subspaces of harmonic polynomials contained in these are denoted

$$\begin{aligned} H(p,q) &:= \mathrm{Harm}_k(\mathbb{H}^d,\mathbb{C}) \cap \mathrm{Hom}_H(p,q), \qquad p+q=k, \\ K(p,q) &:= \mathrm{Harm}_k(\mathbb{H}^d,\mathbb{C}) \cap \mathrm{Hom}_K(p,q), \qquad p+q=k, \\ H_k^{(a,b)}(\mathbb{H}^d) &:= \mathrm{Harm}_k(\mathbb{H}^d,\mathbb{C}) \cap \mathrm{Hom}_k^{(a,b)}(\mathbb{H}^d) \\ &= K(k-a,a) \cap H(k-b,b). \end{aligned}$$

When either p or q above is negative, then we have, by definition, the zero subspace. The subspaces $H(p,q)$ are the irreducible subspaces of $\mathrm{Harm}_k(\mathbb{C}^{2d},\mathbb{C}) \cong \mathrm{Harm}_k(\mathbb{H}^d,\mathbb{C})$ under the action of (left) multiplication by $U(\mathbb{C}^{2d})$, e.g., see [22], from where we borrow the notation $H(p,q)$. Since $U(\mathbb{H}^d)$ is a subgroup of $U(\mathbb{C}^{2d})$, the decomposition of $\mathrm{Harm}_k(\mathbb{H}^d,\mathbb{C})$ into $U(\mathbb{H}^d)$-irreducibles is obtained by decomposing each $H(p,q)$.

The following dimensions are easily calculated

$$\dim_{\mathbb{C}}(\mathrm{Hom}_H(p,q)) = \dim_{\mathbb{C}}(\mathrm{Hom}_K(p,q)) = \binom{p+2d-1}{2d-1}\binom{q+2d-1}{2d-1}, \tag{5.1}$$

$$\dim_{\mathbb{C}}(H(p,q)) = \dim_{\mathbb{C}}(K(p,q)) = (p+q+2d-1)\frac{(p+2d-2)!(q+2d-2)!}{p!q!(2d-1)!(2d-2)!}, \tag{5.2}$$

whilst those of $\mathrm{Hom}_k^{(a,b)}(\mathbb{H}^d)$ and $H_k^{(a,b)}(\mathbb{H}^d)$ are more complicated (Lemmas 7.1 and 7.4).

Example 5.1. Let $H(p,q)^{\mathbb{R}}$ be $H(p,q) \subset \mathrm{Harm}_k(\mathbb{C}^{2d},\mathbb{C})$ viewed as a real vector space, which is invariant under the action of $U(\mathbb{C}^{2d})$. Complexifying this space gives

$$\mathbb{C}H(p,q)^{\mathbb{R}} = H(p,q) \oplus H(q,p).$$

Thus we observe that enlarging the field, which preserves the invariance of subspaces, does not always preserve irreducibility.

Lemma 5.2. *For all integers a and b, we have*

$$\begin{aligned} R\,\mathrm{Hom}_H(a,b) &\subset \mathrm{Hom}_H(a-1,b+1), \quad R^*\,\mathrm{Hom}_H(a,b) \subset \mathrm{Hom}_H(a+1,b-1), \\ L\,\mathrm{Hom}_K(a,b) &\subset \mathrm{Hom}_K(a-1,b+1), \quad L^*\,\mathrm{Hom}_K(a,b) \subset \mathrm{Hom}_K(a+1,b-1), \end{aligned}$$

and $\mathrm{Hom}_K(a,b)$ is invariant under right multiplication by $\mathbb{H}^*$, $\mathrm{Hom}_H(a,b)$ is invariant under left multiplication by $\mathbb{H}^*$, and more generally by $U(\mathbb{H}^d)$, which gives

$$\begin{aligned} R\,\mathrm{Hom}_K(a,b) &\subset \mathrm{Hom}_K(a,b), \quad R^*\,\mathrm{Hom}_K(a,b) \subset \mathrm{Hom}_K(a,b), \\ L\,\mathrm{Hom}_H(a,b) &\subset \mathrm{Hom}_H(a,b), \quad L^*\,\mathrm{Hom}_H(a,b) \subset \mathrm{Hom}_H(a,b). \end{aligned}$$

In particular, for $\alpha,\beta \ge 0$, we have

$$L^\alpha R^\beta\,\mathrm{Hom}_k^{(a,b)}(\mathbb{H}^d,\mathbb{C}) \subset \mathrm{Hom}_k^{(a+\alpha,b+\beta)}(\mathbb{H}^d,\mathbb{C}), \tag{5.3}$$

$$(L^*)^\alpha (R^*)^\beta\,\mathrm{Hom}_k^{(a,b)}(\mathbb{H}^d,\mathbb{C}) \subset \mathrm{Hom}_k^{(a-\alpha,b-\beta)}(\mathbb{H}^d,\mathbb{C}). \tag{5.4}$$

Moreover, for the inner products (4.1) and (4.3) we have the orthogonal direct sums

$$\mathrm{Hom}_H(k-a,a) = (\mathrm{Hom}_H(k-a,a) \cap \ker R^*) \oplus R\,\mathrm{Hom}_H(k-a+1,a-1), \tag{5.5}$$

$$\mathrm{Hom}_H(k-a,a) = (\mathrm{Hom}_H(k-a,a) \cap \ker R) \oplus R^*\,\mathrm{Hom}_H(k-a-1,a+1). \tag{5.6}$$

Proof. The inclusions follow by (elementary) direct calculations.

Let $f \in \operatorname{Hom}_H(k-a,a)$. Since $R^*f \in \operatorname{Hom}_H(k-a+1,a-1)$, we have

$$\begin{aligned} f \in \ker R^* \quad &\iff \quad R^*f = 0 \quad \iff \quad \langle R^*f, g\rangle = 0, \quad \forall g \in \operatorname{Hom}_H(k-a+1,a-1) \\ &\iff \quad \langle f, Rg\rangle = 0, \quad \forall g \in \operatorname{Hom}_H(k-a+1,a-1) \\ &\iff \quad f \in (R \operatorname{Hom}_H(k-a+1,a-1))^\perp, \end{aligned}$$

so that

$$\begin{aligned} \operatorname{Hom}_H(k-a,a) &= (R \operatorname{Hom}_H(k-a+1,a-1))^\perp \oplus R \operatorname{Hom}_H(k-a+1,a-1) \\ &= (\operatorname{Hom}_H(k-a,a) \cap \ker R^*) \oplus R \operatorname{Hom}_H(k-a+1,a-1), \end{aligned}$$

which gives (5.5). The proof of (5.6) is similar. □

By restricting (5.3), (5.4), (5.5), and (5.6) to the harmonic polynomials, we have

$$L^\alpha R^\beta H_k^{(a,b)}(\mathbb{H}^d,\mathbb{C}) \subset H_k^{(a+\alpha,b+\beta)}(\mathbb{H}^d,\mathbb{C}), \tag{5.7}$$

$$(L^*)^\alpha (R^*)^\beta H_k^{(a,b)}(\mathbb{H}^d,\mathbb{C}) \subset H_k^{(a-\alpha,b-\beta)}(\mathbb{H}^d,\mathbb{C}), \tag{5.8}$$

$$H(k-a,a) = (H(k-a,a) \cap \ker R^*) \oplus RH(k-a+1,a-1), \tag{5.9}$$

$$H(k-a,a) = (H(k-a,a) \cap \ker R) \oplus R^*H(k-a-1,a+1). \tag{5.10}$$

Henceforth, all "orthogonal" direct sum decompositions will hold for both the inner products (4.1) and (4.3), unless stated otherwise.

We now give some technical results, related to the commutativity relation (3.16).

Lemma 5.3. *The operators L and L^* commute with R and R^*, and we have*

$$[R^*,R] = R^*R - RR^* = \sum_j \Big(z_j \frac{\partial}{\partial z_j} + w_j \frac{\partial}{\partial w_j} - \overline{z_j} \frac{\partial}{\partial \overline{z_j}} - \overline{w_j} \frac{\partial}{\partial \overline{w_j}} \Big), \tag{5.11}$$

$$[L^*,L] = L^*L - LL^* = \sum_j \Big(z_j \frac{\partial}{\partial z_j} - w_j \frac{\partial}{\partial w_j} - \overline{z_j} \frac{\partial}{\partial \overline{z_j}} + \overline{w_j} \frac{\partial}{\partial \overline{w_j}} \Big). \tag{5.12}$$

Clearly, the right hand side of (5.11) and of (5.12) maps the monomial m_a of (2.10) to a scalar multiple of itself, and so we obtain

$$R^*Rf = RR^*f + (a-b)f, \qquad f \in \operatorname{Hom}_H(a,b), \tag{5.13}$$

$$L^*Lf = LL^*f + (a-b)f, \qquad f \in \operatorname{Hom}_K(a,b). \tag{5.14}$$

We can iterate these to obtain formulas which interchange R and R^*, and L and L^*.

Lemma 5.4. *We have*

$$R^*R^\beta f = R^\beta R^* f + \beta(a-b-\beta+1)R^{\beta-1}f, \quad f \in \operatorname{Hom}_H(a,b), \tag{5.15}$$

$$L^*L^\beta f = L^\beta L^* f + \beta(a-b-\beta+1)L^{\beta-1}f, \quad f \in \operatorname{Hom}_K(a,b), \tag{5.16}$$

which also holds for $\beta = 0$ (in the obvious way).

Here are the most general formulas, which we will use (see [21] for a verification).

Lemma 5.5. *For all choices of α and β, we have*

$$(R^*)^\alpha R^\beta f = \sum_{c=0}^{\alpha} \binom{\alpha}{c} (-\beta)_c (b-a+\beta-\alpha)_c R^{\beta-c} (R^*)^{\alpha-c} f, \quad f \in \mathrm{Hom}_H(a,b), \tag{5.17}$$

$$R^\alpha (R^*)^\beta f = \sum_{c=0}^{\alpha} \binom{\alpha}{c} (-\beta)_c (a-b+\beta-\alpha)_c (R^*)^{\beta-c} R^{\alpha-c} f, \quad f \in \mathrm{Hom}_H(a,b), \tag{5.18}$$

$$(L^*)^\alpha L^\beta f = \sum_{c=0}^{\alpha} \binom{\alpha}{c} (-\beta)_c (b-a+\beta-\alpha)_c L^{\beta-c} (L^*)^{\alpha-c} f, \quad f \in \mathrm{Hom}_K(a,b), \tag{5.19}$$

$$L^\alpha (L^*)^\beta f = \sum_{c=0}^{\alpha} \binom{\alpha}{c} (-\beta)_c (a-b+\beta-\alpha)_c (L^*)^{\beta-c} L^{\alpha-c} f, \quad f \in \mathrm{Hom}_K(a,b). \tag{5.20}$$

Here the terms involving a negative power of an operator have a zero coefficient.

6. $U(\mathbb{H}^d)$-invariant subspaces

It follows from Lemma 5.5 that R and R^* inverses of each other in some sense.

Lemma 6.1. *For $\alpha \le \beta$, and $\beta > \alpha + a - b$ or $\beta \le a - b$, we have*

$$(R^*)^\alpha R^\beta (\ker R^* \cap \mathrm{Hom}_H(a,b)) = R^{\beta-\alpha} (\ker R^* \cap \mathrm{Hom}_H(a,b)), \tag{6.1}$$

$$R^\alpha (R^*)^\beta (\ker R \cap \mathrm{Hom}_H(b,a)) = (R^*)^{\beta-\alpha} (\ker R \cap \mathrm{Hom}_H(b,a)), \tag{6.2}$$

otherwise

$$(R^*)^\alpha R^\beta (\ker R^* \cap \mathrm{Hom}_H(a,b)) = 0, \qquad R^\alpha (R^*)^\beta (\ker R \cap \mathrm{Hom}_H(b,a)) = 0. \tag{6.3}$$

Proof. For $f \in \ker R^* \cap \mathrm{Hom}_H(a,b)$, $R^* f = 0$, and so (5.17) reduces to

$$(R^*)^\alpha R^\beta f = (-\beta)_\alpha (b-a+\beta-\alpha)_\alpha R^{\beta-\alpha} f.$$

The condition for the constant above to be nonzero is $\alpha \le \beta$, and the α factors

$$b-a+\beta-\alpha, \quad b-a+\beta-\alpha+1, \quad \ldots \quad b-a+\beta-1$$

of $(b-a+\beta-\alpha)_\alpha$ are not zero, i.e., $b-a+\beta-\alpha > 0$ or $b-a+\beta-1 < 0$. This gives the first case, with the other following by the same argument. □

By repeated applications of (5.5) and (5.6), we obtain the following.

Lemma 6.2. *We have the orthogonal direct sums*

$$\mathrm{Hom}_H(k-b,b) = \bigoplus_{j=0}^{b} R^{b-j} \big(\ker R^* \cap \mathrm{Hom}_H(k-j,j) \big), \quad b \le k-b, \tag{6.4}$$

$$\mathrm{Hom}_H(k-b,b) = \bigoplus_{j=0}^{k-b} (R^*)^{k-b-j} \big(\ker R \cap \mathrm{Hom}_H(j,k-j) \big), \quad k-b \le b. \tag{6.5}$$

Further

(i) *For* $a > b$, R *is 1-1 on* $\mathrm{Hom}_H(a,b)$.
(ii) *For* $a \le b$, R *maps* $\mathrm{Hom}_H(a,b)$ *onto* $\mathrm{Hom}_H(a-1,b+1)$.

Proof. Apply (5.5) and (5.6) repeatedly. For $b \le k-b$, we have

$$\begin{aligned}\mathrm{Hom}_H(k-b,b) &= (\ker R^* \cap \mathrm{Hom}_H(k-b,b)) \oplus R\,\mathrm{Hom}_H(k-b+1,b-1)\\ &= (\ker R^* \cap \mathrm{Hom}_H(k-b,b))\\ &\qquad \oplus R\{(\ker R^* \cap \mathrm{Hom}_H(k-b+1,b-1)) \oplus R\,\mathrm{Hom}_H(k-b+2,b-2)\}\\ &= (\ker R^* \cap \mathrm{Hom}_H(k-b,b)) \oplus R(\ker R^* \cap \mathrm{Hom}_H(k-b+1,b-1))\\ &\qquad \oplus R^2(\ker R^* \cap \mathrm{Hom}_H(k-b+2,b-2)) \oplus \cdots \oplus R^b(\ker R^* \cap \mathrm{Hom}_H(k,0)).\end{aligned}$$

Similarly, for $k-b \le b$, we have

$$\begin{aligned}\mathrm{Hom}_H(k-b,b) = (\ker R \cap \mathrm{Hom}_H(k-b,b)) &\oplus R^*(\ker R \cap \mathrm{Hom}_H(k-b-1,b+1))\\ &\oplus (R^*)^2(\ker R \cap \mathrm{Hom}_H(k-b-2,b+2)) \oplus \cdots\\ &\cdots \oplus (R^*)^{k-b}(\ker R \cap \mathrm{Hom}_H(0,k)),\end{aligned}$$

which gives (6.4) and (6.5).

To show the injectivity of (i), it suffices to show that for $k-b > b$, i.e., $b+1 \le k-b$, R is 1-1 on each summand in (6.4), i.e.,

$$R^{b-j}\big(\ker R^* \cap \mathrm{Hom}_H(k-j,j)\big), \qquad 0 \le j \le b.$$

This follows from

$$R^*RR^{b-j}\big(\ker R^* \cap \mathrm{Hom}_H(k-j,j)\big) = R^{b-j}\big(\ker R^* \cap \mathrm{Hom}_H(k-j,j)\big),$$

which is (6.1) of Lemma 6.1 for $\alpha = 1$, $\beta = b-j+1$, $a = k-j$, $b = j$, since

$$b+1 \le k-b, \quad j \le b \quad \Longrightarrow \quad b+j+1 \le k, \quad \text{i.e., the condition } \beta \le a-b \text{ holds.}$$

For $a \le b$, a similar argument shows that R^* is 1-1 on $\mathrm{Hom}_H(a-1,b+1)$. Here, when $a=0$, $\mathrm{Hom}_H(a-1,b+1) = 0$. Therefore, $(R^*|_{\mathrm{Hom}_H(a-1,b+1)})^* = R|_{\mathrm{Hom}_H(a,b)}$ is onto, and we have (ii). □

The following result says that the j-terms in the expansions of Lemma 6.2 (only one of which holds for a given b, $2b \ne k$) are in fact equal. This then allows for a single expansion for both cases (Lemma 6.5).

Lemma 6.3. *(Row movements) For* $0 \le j \le \frac{k}{2}$, *we have*

$$R^{k-2j+1}(\ker R^* \cap \mathrm{Hom}_H(k-j,j)) = 0, \tag{6.6}$$

$$(R^*)^{k-2j+1}(\ker R \cap \mathrm{Hom}_H(j,k-j)) = 0, \tag{6.7}$$

and for $j \le a \le k-j$, *we have*

$$R^{a-j}(\ker R^* \cap \mathrm{Hom}_H(k-j,j)) = (R^*)^{k-a-j}(\ker R \cap \mathrm{Hom}_H(j,k-j)). \tag{6.8}$$

Furthermore, all of the results above hold with $\mathrm{Hom}_H(p,q)$ *replaced by* $H(p,q)$.

Proof. The equations (6.6) and (6.7) follow from Lemma 6.1 for the choice $\alpha = 0$, $\beta = k-2j+1$, $a = k-j$, $b = j$. These give the inclusions

$$R^{k-2j}(\ker R^* \cap \mathrm{Hom}_H(k-j,j)) \subset \ker R \cap \mathrm{Hom}_H(j,k-j),$$
$$(R^*)^{k-2j}(\ker R \cap \mathrm{Hom}_H(j,k-j)) \subset \ker R^* \cap \mathrm{Hom}_H(k-j,j).$$

We now prove the cases $a = k-j$ and $a = j$ in (6.8), i.e.,

$$R^{k-2j}(\ker R^* \cap \mathrm{Hom}_H(k-j,j)) = \ker R \cap \mathrm{Hom}_H(j,k-j), \tag{6.9}$$
$$(R^*)^{k-2j}(\ker R \cap \mathrm{Hom}_H(j,k-j)) = \ker R^* \cap \mathrm{Hom}_H(k-j,j). \tag{6.10}$$

Taking $\alpha = \beta = k-2j$ in Lemma 6.1 gives

$$(R^*)^{k-2j}R^{k-2j}(\ker R^* \cap \mathrm{Hom}_H(k-j,j)) = \ker R^* \cap \mathrm{Hom}_H(k-j,j), \tag{6.11}$$
$$R^{k-2j}(R^*)^{k-2j}(\ker R \cap \mathrm{Hom}_H(j,k-j)) = \ker R \cap \mathrm{Hom}_H(j,k-j).$$

Thus we have

$$\begin{aligned}\ker R \cap \mathrm{Hom}_H(j,k-j) &= R^{k-2j}(R^*)^{k-2j}(\ker R \cap \mathrm{Hom}_H(j,k-j)) \\ &\subset R^{k-2j}(\ker R^* \cap \mathrm{Hom}_H(k-j,j)) \\ &\subset \ker R \cap \mathrm{Hom}_H(j,k-j),\end{aligned}$$

which gives (6.9), with (6.10) following similarly. Now (6.11) and (6.9) give

$$\begin{aligned}R^{a-j}(\ker R^* \cap \mathrm{Hom}_H(k-j,j)) &= R^{a-j}(R^*)^{k-2j}R^{k-2j}(\ker R^* \cap \mathrm{Hom}_H(k-j,j)) \\ &= R^{a-j}(R^*)^{k-2j}(\ker R \cap \mathrm{Hom}_H(j,k-j)).\end{aligned} \tag{6.12}$$

Taking $\alpha = a-j$, $\beta = k-2j$ in Lemma 6.1 gives

$$R^{a-j}(R^*)^{k-2j}(\ker R \cap \mathrm{Hom}_H(j,k-j)) = (R^*)^{k-j-a}(\ker R \cap \mathrm{Hom}_H(j,k-j)),$$

which together with (6.12) gives (6.8). $\square$

Combining Lemma 6.3 and its counter part for L gives the following.

Lemma 6.4. *(Square array) For $0 \le j \le \frac{k}{2}$, $j \le a, b \le k-j$, we have*

$$\begin{aligned}L^{a-j}R^{b-j}(\ker L^* \cap \ker R^* \cap H_k^{(j,j)}) &= (L^*)^{k-a-j}(R^*)^{k-b-j}(\ker L \cap \ker R \cap H_k^{(k-j,k-j)}) \\ &= L^{a-j}(R^*)^{k-b-j}(\ker L^* \cap \ker R \cap H_k^{(j,k-j)}) \\ &= (L^*)^{k-a-j}R^{b-j}(\ker L \cap \ker R^* \cap H_k^{(k-j,j)}),\end{aligned}$$

with

$$\dim(L^{a-j}R^{b-j}(\ker L^* \cap \ker R^* \cap H_k^{(j,j)})) = \dim(\ker L^* \cap \ker R^* \cap H_k^{(j,j)}).$$

We now present a key technical result.

Lemma 6.5. *We have the orthogonal direct sum decompositions*

$$\operatorname{Hom}_k(\mathbb{H}^d,\mathbb{C}) = \bigoplus_{0\le j\le \frac{k}{2}} \bigoplus_{j\le b\le k-j} \operatorname{Hom}_H(k-b,b)_{k-2j}, \tag{6.13}$$

$$\operatorname{Harm}_k(\mathbb{H}^d,\mathbb{C}) = \bigoplus_{0\le j\le \frac{k}{2}} \bigoplus_{j\le b\le k-j} H(k-b,b)_{k-2j}, \tag{6.14}$$

into $U(\mathbb{H}^d)$-invariant subspaces, where

$$\begin{aligned} \operatorname{Hom}_H(k-b,b)_{k-2j} &:= R^{b-j}\big(\ker R^* \cap \operatorname{Hom}_H(k-j,j)\big) \\ &= (R^*)^{k-b-j}\big(\ker R \cap \operatorname{Hom}_H(j,k-j)\big) \\ &\subset \operatorname{Hom}_H(k-b,b), \end{aligned} \tag{6.15}$$

$$\begin{aligned} H(k-b,b)_{k-2j} &:= R^{b-j}\big(\ker R^* \cap H(k-j,j)\big) \\ &= (R^*)^{k-b-j}\big(\ker R \cap H(j,k-j)\big) \\ &\subset H(k-b,b). \end{aligned} \tag{6.16}$$

Proof. Since the Laplacian operator Δ commutes with R and R^* (Lemma 4.3), the decomposition (6.14) follows from (6.13) by taking the intersection with the harmonic polynomials. We therefore consider just the decomposition of $\operatorname{Hom}_H(k-b,b)$.

Since $\operatorname{Hom}_H(k-b,b)$ and $H(k-b,b)$ are invariant under $U(\mathbb{C}^{2d})$, they are invariant under $U(\mathbb{H}^d)$. Moreover, the action of $U(\mathbb{H}^d)$ commutes with R and R^* (Lemma 4.4), and so the summands in (6.13) and (6.14) are $U(\mathbb{H}^d)$-invariant. As an indicative calculation, let $U \in U(\mathbb{H}^d)$, then

$$f \in \ker R^* \iff U\cdot(R^*f)=0 \iff R^*(U\cdot f)=0 \iff U\cdot f\in\ker R^*,$$

and so

$$\begin{aligned} U\cdot H(k-b,b)_{k-2j} &= U\cdot R^{b-j}\big(\ker R^*\cap H(k-j,j)\big) = R^{b-j}\big(U\cdot\ker R^*\cap U\cdot H(k-j,j)\big) \\ &= R^{b-j}\big(\ker R^*\cap H(k-j,j)\big) = H(k-b,b)_{k-2j}. \end{aligned}$$

Since

$$j\le b\le k-j \iff j\le b,\quad j\le k-b \iff j\le\min\{b,k-b\},$$

the direct sum (6.13) can be rearranged as

$$\operatorname{Hom}_k(\mathbb{H}^d,\mathbb{C}) = \bigoplus_{0\le b\le k} \bigoplus_{j=0}^{\min\{b,k-b\}} \operatorname{Hom}_H(k-b,b)_{k-2j}.$$

By Lemma 6.3,

$$R^{b-j}\big(\ker R^*\cap\operatorname{Hom}_H(k-j,j)\big) = (R^*)^{k-b-j}\big(\ker R\cap\operatorname{Hom}_H(j,k-j)\big),$$

which gives the equalities in (6.15) and (6.16), and so it suffices to show the orthogonal direct sums

$$\mathrm{Hom}_H(k-b,b)=\bigoplus_{j=0}^{b}R^{b-j}\big(\ker R^*\cap \mathrm{Hom}_H(k-j,j)\big),\quad b\le k-b,$$

$$\mathrm{Hom}_H(k-b,b)=\bigoplus_{j=0}^{k-b}(R^*)^{k-b-j}\big(\ker R\cap \mathrm{Hom}_H(j,k-j)\big),\quad k-b\le b.$$

These are given by Lemma 6.2. $\square$

To calculate the dimensions of various irreducibles, we will need the following.

Lemma 6.6. *Let $0\le j\le \frac{k}{2}$. For $d=1$, we have the following dimensions*

$$\dim\big(\ker R^*\cap \mathrm{Hom}_H(k-j,j)\big)=k-2j+1,\quad \dim\big(\ker R^*\cap H(k-j,j)\big)=\begin{cases}1, & j=0;\\ 0, & j\ne 0.\end{cases}$$

For $d\ge 2$, we have

$$\dim\big(\ker R^*\cap \mathrm{Hom}_H(k-j,j)\big)=(k-2j+1)\frac{(k-j+2d-1)!(j+2d-2)!}{(k-j+1)!j!(2d-1)!(2d-2)!},\tag{6.17}$$

$$\dim\big(\ker R^*\cap H(k-j,j)\big)=(k-2j+1)(k+2d-1)\frac{(k-j+2d-2)!(j+2d-3)!}{(k-j+1)!j!(2d-1)!(2d-3)!}.\tag{6.18}$$

Proof. From (5.5), and the fact R is 1-1 on $\mathrm{Hom}_H(k-j+1,j-1)$ (Lemma 6.2), we have

$$\begin{aligned}\dim(\mathrm{Hom}_H(k-j,j)\cap\ker R^*)&=\dim\left(\mathrm{Hom}_H(k-j,j)\right)-\dim\left(R\,\mathrm{Hom}_H(k-j+1,j-1)\right)\\&=\dim\left(\mathrm{Hom}_H(k-j,j)\right)-\dim\left(\mathrm{Hom}_H(k-j+1,j-1)\right).\end{aligned}$$

Using this and (5.1), with $p=k-j$, $q=j$, we calculate (6.17) for $d\ge 1$

$$\begin{aligned}&\dim(\ker R^*\cap \mathrm{Hom}_H(k-j,j))\\&\quad=\frac{1}{(2d-1)!^2}\left\{\frac{(p+2d-1)!(q+2d-1)!}{p!q!}-\frac{(p+2d)!(q+2d-2)!}{(p+1)!(q-1)!}\right\}\\&\quad=\frac{(p+2d-1)!(q+2d-2)!}{(p+1)!q!(2d-1)!^2}\{(q+2d-1)(p+1)-(p+2d)q\}\\&\quad=\frac{(p+2d-1)!(q+2d-2)!}{(p+1)!q!(2d-1)!^2}(p-q+1)(2d-1).\end{aligned}$$

The other formula follows in a similar way, from

$$\begin{aligned}\dim(\ker R^*\cap H(k-j,j))&=\dim(H(k-j,j))-\dim(H(k-j+1,j-1))\\&=\binom{k-j+2d-1}{2d-1}\binom{j+2d-1}{2d-1}-\binom{k-j+2d}{2d-1}\binom{j+2d-2}{2d-1},\end{aligned}$$

with the $d=1$ case calculated separately, which completes the proof. $\square$

Example 6.7. For $j=0$, (6.18) reduces to

$$\dim\big(\ker R^*\cap H(k,0)\big)=\frac{(k+2d-1)!}{k!(2d-1)!}=\dim\big(H(k,0)\big),$$

so that $\ker R^* \cap H(k,0)$ is the holomorphic polynomials, i.e.,

$$\ker R^* \cap H(k,0) = H(k,0) = \bigoplus_{|\alpha+\beta|=k} \operatorname{span}\{z^\alpha w^\beta\} \quad \text{(orthogonal direct sum)}.$$

We also observe, from the proof of Lemma 6.6, that for $0 \leq j \leq \frac{k}{2}$, we have

$$\ker R^* \cap H(k-j,j) = \{f \in H(k-j,j) : f \perp \bigoplus_{0\leq a<j} H(k-a,a)\},$$

so the by applying Gram-Schmidt to a spanning sequence ordered so that its elements are in $H(k,0), H(k-1,1), \ldots H(k-j,j)$, successively, the corresponding elements are an orthonormal basis for $\ker R^* \cap H(k,0), \ldots, \ker R^* \cap H(k-j,j)$.

The results of this section can found or deduced from those of the work of [4]. Their variables $z_1, \ldots, z_{2p}$ correspond to ours via

$$z_1, \ldots, z_{2p} \quad \longleftrightarrow \quad z_1, w_1, \ldots, z_p, w_p,$$

and they define operators

$$\varepsilon = R^*, \quad \varepsilon^\dagger = R.$$

The decomposition (6.14) for $H(k-b,b)$ of Lemma 6.5 is presented as the two cases in Lemma 6.2 (Theorems 5.1 and 5.2 of §5 [4]).

7. Visualising the action of L and R on subspaces

The action of L and R given in Lemma 5.2 leads to the following orthogonal direct sums.

Lemma 7.1. *We have the orthogonal direct sum decomposition into $(k+1)^2$ subspaces*

$$\operatorname{Hom}_k(\mathbb{H}^d, \mathbb{C}) = \bigoplus_{0\leq a,b\leq k} \operatorname{Hom}_k^{(a,b)}(\mathbb{H}^d). \tag{7.1}$$

For $0 \leq a,b \leq k$, let $m_a = m_a^{(k)} := \min\{a, k-a\}$, $m_b = m_b^{(k)} := \min\{b, k-b\}$, and

$$m = m_{a,b}^{(k)} := \min\{m_a, m_b\}, \quad M = M_{a,b}^{(k)} := \max\{m_a, m_b\}, \quad c := \min\{a,b\}. \tag{7.2}$$

Then

$$\operatorname{Hom}_k^{(a,b)}(\mathbb{H}^d) = \operatorname{span}\{z^{\alpha_1} w^{\alpha_2} \overline{z}^{\alpha_3} \overline{w}^{\alpha_4}\}_{(\alpha_1,\alpha_2,\alpha_3,\alpha_4)\in A}, \tag{7.3}$$

where $A = A_{a,b,k}$ is given by

$$A := \{\alpha : |\alpha_1| = k-(a+b-c)-j, |\alpha_2| = a-c+j, |\alpha_3| = c-j, |\alpha_4| = b-c+j, 0 \leq j \leq m\}.$$

In particular, we have

$$\dim\big(\mathrm{Hom}_k^{(a,b)}(\mathbb{H}^d,\mathbb{C})\big) = \sum_{j=0}^{m} \binom{k-M-j+d-1}{d-1}\binom{j+d-1}{d-1} \times \binom{m-j+d-1}{d-1}\binom{M-m+j+d-1}{d-1}. \tag{7.4}$$

Proof. To establish (7.1), it suffices to show that the direct sums

$$\mathrm{Hom}_k(\mathbb{H}^d,\mathbb{C}) = \bigoplus_{a+b=k} \mathrm{Hom}_H(a,b) = \bigoplus_{p+q=k} \mathrm{Hom}_K(p,q),$$

are orthogonal, which follows immediately since the monomials are orthogonal for (4.3).

We now consider (7.3). Let $f = z^{\alpha_1} w^{\alpha_2} \overline{z}^{\alpha_3} \overline{w}^{\alpha_4} \in \mathrm{Hom}_k^{(a,b)}(\mathbb{H}^d,\mathbb{C})$, i.e.,

$$|\alpha_1| + |\alpha_4| = k-a, \quad |\alpha_2| + |\alpha_3| = a, \qquad |\alpha_1| + |\alpha_2| = k-b, \quad |\alpha_3| + |\alpha_4| = b.$$

The above equations imply that once an allowable value of $|\alpha_1|, |\alpha_2|, |\alpha_3|, |\alpha_4|$ is specified, then the others are uniquely determined. The allowable values are determined by an equation where the right hand side is $m = \min\{a, k-a, b, k-b\}$, and so we must treat (four) cases. First consider the case $a \le b$, i.e., $m = a, k-b$, for which we have

$$0 \le j = |\alpha_2| \le m \in \{a, k-b\}, \quad |\alpha_3| = a-j, \quad |\alpha_1| = k-b-j, \quad |\alpha_4| = j+b-a,$$

and hence

$$\mathrm{Hom}_k^{(a,b)}(\mathbb{H}^d,\mathbb{C}) = \mathrm{span}\{z^{\alpha_1} w^{\alpha_2} \overline{z}^{\alpha_3} \overline{w}^{\alpha_4} : |\alpha_1| = k-b-j, |\alpha_2| = j, |\alpha_3| = a-j, |\alpha_4| = j+b-a, 0 \le j \le m\}.$$

The corresponding condition for the case $a \ge b$ is

$$0 \le j = |\alpha_4| \le m \in \{b, k-a\}, \quad |\alpha_1| = k-a-j, \quad |\alpha_2| = a-b+j, \quad |\alpha_3| = b-j,$$

and so we obtain (7.3).

It follows from symmetries of the space $\mathrm{Hom}_k^{(a,b)}(\mathbb{H}^d,\mathbb{C})$, or by direct calculation, that its dimension, the cardinality of A, depends only on m, M (and k). Therefore, by the case $m = a = c$, $M = b$, and the fact $\alpha_1, \ldots, \alpha_4 \in \mathbb{Z}_+^d$, we obtain (7.4). $\square$

Example 7.2. For $d = 1$, we have

$$\dim(\mathrm{Hom}_k^{(a,b)}(\mathbb{H})) = m+1, \qquad m := \min\{a, k-a, b, k-b\}. \tag{7.5}$$

Example 7.3. For $k = 1$, we have

$$\mathrm{Hom}_1^{(0,0)}(\mathbb{H}^d) = \mathrm{span}\{z_1, \ldots, z_d\}, \qquad \mathrm{Hom}_1^{(0,1)}(\mathbb{H}^d) = \mathrm{span}\{\overline{w_1}, \ldots, \overline{w_d}\},$$
$$\mathrm{Hom}_1^{(1,0)}(\mathbb{H}^d) = \mathrm{span}\{w_1, \ldots, w_d\}, \qquad \mathrm{Hom}_1^{(1,1)}(\mathbb{H}^d) = \mathrm{span}\{\overline{z_1}, \ldots, \overline{z_d}\}.$$

The corresponding result for harmonic polynomials is the following (see [21]).

Lemma 7.4. *We have the orthogonal direct sum decomposition into $(k+1)^2$ subspaces*

$$\mathrm{Harm}_k(\mathbb{H}^d,\mathbb{C}) = \bigoplus_{0\le a,b\le k} H_k^{(a,b)}(\mathbb{H}^d), \tag{7.6}$$

where

$$\dim(H_k^{(a,b)}(\mathbb{H}^d)) = \dim(\mathrm{Hom}_k^{(a,b)}(\mathbb{H}^d)) - \dim(\mathrm{Hom}_{k-2}^{(a-1,b-1)}(\mathbb{H}^d)). \tag{7.7}$$

In particular, for $d=1$, we have

$$\dim(H_k^{(a,b)}(\mathbb{H})) = 1, \qquad 0\le a,b\le k, \tag{7.8}$$

and for $d>1$, with m and M given by (7.2), we have

$$\dim(H_k^{(a,b)}(\mathbb{H}^d,\mathbb{C})) = F(k,m,M,d), \qquad 0\le a,b\le k, \tag{7.9}$$

where

$$F(k,m,M,d) := \sum_{j=0}^{m}\binom{j+d-1}{d-1}\binom{M-m+j+d-1}{d-1} \times \frac{(m-j+1)_{d-2}(k-M-j+1)_{d-2}}{(d-1)!(d-2)!}(k-M+m-2j+d-1). \tag{7.10}$$

Example 7.5. For $k=2$, we have three cases $(m,M)=(0,0),(0,1),(1,1)$, giving

$$\dim(H_2^{(a,b)}(\mathbb{H}^d)) = F(2,0,0,d) = \frac{1}{2}d(d+1), \qquad (a,b)\in\{(0,0),(2,0),(0,2),(2,2)\},$$
$$\dim(H_2^{(a,b)}(\mathbb{H}^d)) = F(2,0,1,d) = d^2, \qquad (a,b)\in\{(1,0),(0,1),(1,2),(2,1)\},$$
$$\dim(H_2^{(a,b)}(\mathbb{H}^d)) = F(2,1,1,d) = 2d^2-1, \qquad (a,b)\in\{(1,1)\}.$$

These formulas also hold for $d=1$. We also have

$$\dim(H_k^{(a,b)}(\mathbb{H}^d)) = \binom{k+d-1}{d-1} = \dim(\mathrm{Hom}_k^{(a,b)}(\mathbb{H}^d)), \quad (a,b)\in\{(0,0),(k,0),(0,k),(k,k)\},$$

with the corresponding spaces given by $A=A_{a,b,k}$ of Lemma 7.1, e.g.,

$$H_k^{(0,0)}(\mathbb{H}^d) = \bigoplus_{|\alpha|=k}\mathrm{span}\{z^\alpha\}, \quad \text{(orthogonal direct sum)}.$$

The results of this section for $\mathrm{Harm}_k(\mathbb{H}^d,\mathbb{C})$ can be summarised as follows.

Schematic 7.6. *The orthogonal decomposition (7.6) of* $\mathrm{Harm}_k(\mathbb{H}^d,\mathbb{C})$ *can be displayed as a square matrix/array/table*

$$\begin{array}{c} \\ K(k,0) \\ K(k-1,1) \\ \vdots \\ K(0,k) \end{array} \begin{array}{c} \begin{array}{cccc} H(k,0) & H(k-1,1) & \cdots & H(0,k) \end{array} \\ \begin{bmatrix} H_k^{(0,0)}(\mathbb{H}^d) & H_k^{(0,1)}(\mathbb{H}^d) & \cdots & H_k^{(0,k)}(\mathbb{H}^d) \\ H_k^{(1,0)}(\mathbb{H}^d) & H_k^{(1,1)}(\mathbb{H}^d) & \cdots & H_k^{(1,k)}(\mathbb{H}^d) \\ \vdots & \vdots & & \vdots \\ H_k^{(k,0)}(\mathbb{H}^d) & H_k^{(k,1)}(\mathbb{H}^d) & \cdots & H_k^{(k,k)}(\mathbb{H}^d) \end{bmatrix} \end{array} \qquad \begin{array}{ccc} & L^*\uparrow & \\ \overset{R^*}{\longleftarrow} & & \overset{R}{\longrightarrow} \\ & L\downarrow & \end{array} \tag{7.11}$$

where the rows are indexed by $K(k-a,a)$ and the columns by $H(k-b,b)$. Here

- $\mathrm{Harm}_k(\mathbb{H}^d,\mathbb{C})$ *is the orthogonal direct sum of the $(k+1)^2$ entries of the matrix.*
- *The subspace in the $K(k-a,a)$ row and $H(k-b,b)$ column is*

$$H_k^{(a,b)}(\mathbb{H}^d) = K(k-a,a)\cap H(k-b,b).$$

- $H(k-a,b)$ *is the orthogonal direct sum of the entries of its column.*
- $K(k-a,a)$ *is the orthogonal direct sum of the entries of its row.*
- *Multiplication by L moves down the columns, and L^* up them.*
- *Multiplication by R moves right along the rows, and R^* to the left of them. Further, multiplication by R is 1-1 on the left hand side (half) of the table, and is onto on the right hand side.*
- *Left multiplication by $\mathbb{H}^*$ (and more generally by $U(\mathbb{H}^d)$) moves within the columns.*
- *Right multiplication by $\mathbb{H}^*$ moves within the rows.*
- *Multiplication by L_α, $L_{\overline{\alpha}}$, R_α and $R_{\overline{\alpha}}$ does not move the entries of the matrix.*

There is a similar “square” for the decomposition (7.1) of $\mathrm{Hom}_k(\mathbb{H}^d,\mathbb{C})$.

There are also “symmetries” which permute the entries of the square, such as

$$\overline{H_k^{(a,b)}(\mathbb{H}^d)} = H_k^{(k-a,k-b)}(\mathbb{H}^d).$$

It is convenient to imagine zero subspaces outside of the square matrix, which then encodes properties such as

$$R^{k+1}\,\mathrm{Harm}_k(\mathbb{H}^d,\mathbb{C})=0, \qquad R^*H(k,0)=0, \qquad L^2K(1,k-1)=0.$$

8. The one variable case

We now consider the $d=1$ case in great detail. Though this is somewhat degenerate, and usually not considered, it provides motivation and illustrates the main features of the general case.

By Lemma 7.4, $\dim(\mathrm{Harm}_k(\mathbb{H},\mathbb{C})) = (k+1)^2$, and the square matrix/table (7.11) for $\mathrm{Harm}_k(\mathbb{H},\mathbb{C})$ consists of the one-dimensional subspaces $\{H_k^{(a,b)}(\mathbb{H})\}_{0\le a,b\le k}$. Since the polynomial z^k is holomorphic, it is harmonic, and so

$$H_k^{(0,0)}(\mathbb{H}) = \mathrm{span}_{\mathbb{C}}\{z^k\}.$$

We consider what are the other harmonic monomials in $\mathrm{Hom}_k(\mathbb{H},\mathbb{C})$.

Example 8.1. The monomial $z^{\alpha_1}w^{\alpha_2}\overline{z}^{\alpha_3}\overline{w}^{\alpha_4}$, $|\alpha|=k$, is harmonic if and only if

$$\Big(\frac{\partial^2}{\partial\overline{z}\partial z}+\frac{\partial^2}{\partial\overline{w}\partial w}\Big) z^{\alpha_1}w^{\alpha_2}\overline{z}^{\alpha_3}\overline{w}^{\alpha_4} = \alpha_1\alpha_3 z^{\alpha_1-1}w^{\alpha_2}\overline{z}^{\alpha_3-1}\overline{w}^{\alpha_4} + \alpha_2\alpha_4 z^{\alpha_1}w^{\alpha_2-1}\overline{z}^{\alpha_3}\overline{w}^{\alpha_4-1}=0,$$

i.e., $\alpha_1\alpha_3=\alpha_2\alpha_4=0$. This gives $4k$ harmonic monomials of degree k.

The harmonic monomials of degree k lie on the four “edges” (of length $k+1$) of the square table, which are given by $\alpha_2=\alpha_3=0$ (top edge), $\alpha_3=\alpha_4=0$ (left edge), $\alpha_1=\alpha_2=0$ (right edge), $\alpha_1=\alpha_4=0$ (bottom edge), with

$$H_k^{(\alpha_2+\alpha_3,\alpha_3+\alpha_4)}(\mathbb{H}) = \mathrm{span}_{\mathbb{C}}\{z^{\alpha_1}w^{\alpha_2}\overline{z}^{\alpha_3}\overline{w}^{\alpha_4}\}, \qquad |\alpha|=k, \quad \alpha_1\alpha_3=\alpha_2\alpha_4=0.$$

An elementary calculation shows that

$$L^k R^k(z^k) = (-1)^k k!^2 \overline{z}^k, \tag{8.1}$$

and so it follows from the Schematic 7.6, that by applying L and R to the upper left corner $z^k \in H_k^{(0,0)}(\mathbb{H})$, that we can "fill out the table" with nonzero polynomials in the subspaces, which (in this case) gives a basis for them, e.g., for $k = 2$, we have

$$\begin{array}{ccc} z^2 \xrightarrow{R} & & \\ L\downarrow & & \end{array} \quad \begin{array}{c} \\ K(2,0) \\ K(1,1) \\ K(0,2) \end{array} \begin{array}{c} \begin{array}{ccc} H(2,0) & H(1,1) & H(0,2) \end{array} \\ \begin{bmatrix} z^2 & 2z\overline{w} & 2\overline{w}^2 \\ 2zw & 2w\overline{w} - 2z\overline{z} & -4\overline{z}\overline{w} \\ 2w^2 & -4\overline{z}w & 4\overline{z}^2 \end{bmatrix} \end{array}. \tag{8.2}$$

Since L and R commute, it makes no difference how one fills out the table by applying L and R, e.g., the middle entry can be obtained as either of

$$RL(z^2) = R(2zw) = 2w\overline{w} - 2z\overline{z}, \qquad LR(z^2) = L(2z\overline{w}) = 2w\overline{w} - 2z\overline{z}.$$

Even in this simple example, one can observe the following features of the general case:

- The harmonic functions on the edges of the square have the simplest description, with the formulas becoming more complicated as one moves towards the centre.
- There are symmetries of the polynomials given by certain permutations of $z, w, \overline{z}, \overline{w}$.
- One can move around the square table by applying L and L^* (down and up) and R and R^* (across and back).

We now show that $L^a R^b(z^k)$ has an increasingly complicated formula as one moves towards the centre of the table.

Lemma 8.2. *The unique harmonic polynomial $p_k^{(a,b)}$ in $H_k^{(a,b)}(\mathbb{H})$ is given by*

$$\begin{aligned} p_k^{(a,b)} &= \sum_{j=0}^{m} \frac{(-1)^j}{j!} \frac{(-c)_j (a+b-c-k)_j}{(|b-a|+1)_j} z^{k-(a+b-c)-j} w^{a-c+j} \overline{z}^{c-j} \overline{w}^{b-c+j} \\ &= \frac{(k-a-b+c)!}{k!(c-a-b)_c} L^a R^b(z^k), \end{aligned} \tag{8.3}$$

where

$$m = \min\{a, b, k-a, k-b\}, \quad c = \min\{a, b\} = \frac{1}{2}(a + b - |b - a|).$$

Proof. We consider the case $a \le b$, i.e., $m = \min\{a, k - b\}$, the other being similar.

By Lemma 7.4, there is a unique (up to a scalar multiple) harmonic polynomial in $\mathrm{Hom}_k^{(a,b)}(\mathbb{H}, \mathbb{C})$, which by (7.3) has the form

$$f = \sum_{j=0}^{m} c_j z^{k-b-j} w^j \overline{z}^{a-j} \overline{w}^{b-a+j}.$$

The condition that f be harmonic, i.e., $\nabla f = 0$, gives

$$\sum_{j=0}^{m-1} c_j(k-b-j)(a-j)z^{k-b-j-1}w^j\overline{z}^{a-j-1}\overline{w}^{b-a+j}$$
$$+\sum_{j=0}^{m-1} c_{j+1}(j+1)(b-a+j+1)z^{k-b-j-1}w^j\overline{z}^{a-j-1}\overline{w}^{b-a+j}=0,$$

and equating coefficients of the monomials gives

$$c_j(k-b-j)(a-j)+c_{j+1}(j+1)(b-a+j+1)=0, \qquad 0\le j\le m-1,$$

so that

$$c_{j+1}=-c_j\frac{(k-b-j)(a-j)}{(j+1)(b-a+j+1)} \quad\Longrightarrow\quad c_j=\frac{(-1)^j}{j!}\frac{(k-b+1-j)_j(a+1-j)_j}{(b-a+1)_j}c_0,$$

which gives the desired formula. □

The indices $\{(a,b)\}_{0\le a,b\le k}$ for the polynomials $p_k^{(a,b)}\in H_k^{(a,b)}(\mathbb{H})$ in the square table can be partitioned into nested squares

$$S_m:=\{(a,b):\min\{a,b,k-a,k-b\}=m\}, \qquad 0\le m\le \frac{k}{2}, \tag{8.4}$$

with S_0 giving the "edges of the table". These have size

$$|S_m|=\begin{cases}4(k-2m), & 0\le m<\frac{k}{2};\\ 1, & m=\frac{k}{2}.\end{cases} \tag{8.5}$$

Here is an illustration for the case $k=4$

$$\begin{matrix}\boxed{0}&\boxed{0}&\boxed{0}&\boxed{0}&\boxed{0}\\ \boxed{0}&\boxed{1}&\boxed{1}&\boxed{1}&\boxed{0}\\ \boxed{0}&\boxed{1}&\boxed{2}&\boxed{1}&\boxed{0}\\ \boxed{0}&\boxed{1}&\boxed{1}&\boxed{1}&\boxed{0}\\ \boxed{0}&\boxed{0}&\boxed{0}&\boxed{0}&\boxed{0}\end{matrix}, \qquad S_0=\{\boxed{0}\}, \quad S_1=\{\boxed{1}\}, \quad S_2=\{\boxed{2}\}.$$

From the formula (8.3), we have the first instance of a general phenomenon:

- The polynomials $p_k^{(a,b)}$, $(a,b)\in S_m$, have $m+1$ terms, i.e., the complexity of the formula for $p_k^{(a,b)}$ increases as one gets closer to the centre of the square array.

We now consider the decomposition of $\mathrm{Harm}_k(\mathbb{H}^d,\mathbb{C})$ into irreducibles.

The irreducible representations of the simply connected compact nonabelian Lie group $\mathrm{Sp}(1)=U(\mathbb{H})=SU(2)$ are well known [14]. For now, we need only that there is precisely one irreducible representation W_k of dimension $k+1$, for each $k\ge 0$.

Theorem 8.3. *For left multiplication by* $\mathbb{H}^*$, *equivalently* $\mathrm{Sp}(1)$, *we have the following orthogonal direct sum of irreducibles*

$$\mathrm{Harm}_k(\mathbb{H},\mathbb{C})=\bigoplus_{0\le a\le k} H(k-a,a)\cong(k+1)W_k,$$

and for right multiplication by $\mathbb{H}^*$*, we have the direct sum of irreducibles*

$$\mathrm{Harm}_k(\mathbb{H},\mathbb{C}) = \bigoplus_{0\le b\le k} K(k-b,b) \cong (k+1)W_k.$$

Proof. From the Schematic 7.6 for $\mathrm{Harm}_k(\mathbb{H},\mathbb{C})$, it follows that by taking columns (respectively rows) of the table gives an orthogonal direct sum of invariant subspaces for action given by left (respectively right) multiplication by $\mathbb{H}^*$ (Lemma 3.8), and so it remains only to show that these $(k+1)$-dimensional subspaces are irreducible.

We now show $K(k-a,a)$ is irreducible for the action given by right multiplication. The other case is similar, and can be found in [12] Theorem 5.37. We have

$$K(k-a,a) = \mathrm{span}_{\mathbb{C}}\{p_k^{(a,b)}\}_{0\le b\le k} = \mathrm{span}\{R^b p_k^{(a,0)}\}_{0\le b\le k}.$$

Consider a nonzero polynomial

$$f = \sum_{0\le b\le b^*} c_b p_k^{(a,b)} \in K(k-a,a), \qquad c_{b^*}\ne 0.$$

Let V be an invariant subspace of $K(k-a,a)$ containing f. Since R^* maps nonzero polynomials left across the table, we have that $(R^*)^{b^*} f$ is a nonzero multiple of $p_k^{(a,0)}$. Thus, V contains $p_k^{(a,0)}$, and hence $Rp_k^{(a,0)},\ldots,R^k p_k^{(a,0)}$, giving $V = K(k-a,a)$, i.e., V is irreducible. □

In both cases, there is a single homogeneous component corresponding to W_k.

Example 8.4. For left multiplication by $\mathbb{H}^*$, i.e., the action given by

$$(\alpha + j\beta)(z+jw) = (\alpha z - \overline{\beta} w) + j(\overline{\alpha} w + \beta z),$$

we have the irreducible representation

$$H(k,0) = \mathrm{span}_{\mathbb{C}}\{z^k, z^{k-1}w, z^{k-2}w^2, \ldots, w^k\},$$

which is given by Folland [12] for the action of $SU(2)\subset \mathbb{H}^*$ given by

$$\begin{pmatrix} \alpha & -\overline{\beta} \\ \beta & \overline{\alpha}\end{pmatrix}\begin{pmatrix} z \\ w\end{pmatrix} = \begin{pmatrix} \alpha z - \overline{\beta} w \\ \overline{\alpha} w + \beta z\end{pmatrix}.$$

Here L and L^* reduce to $L = w\frac{\partial}{\partial z}$ and $L^* = z\frac{\partial}{\partial w}$.

We now consider the combined action given by both left and right multiplication by $\mathbb{H}^*$, i.e., the action of $\mathrm{Sp}(1)\times\mathrm{Sp}(1)$, equivalently $\mathbb{H}^*\times\mathbb{H}^*$, given by

$$\big((q_1,q_2)\cdot f\big)(q) := f(q_1 q\overline{q_2}).$$

The invariant subspaces for this action are invariant under both L and R (and their adjoints). This leads to the following.

Theorem 8.5. *The action of* $\mathrm{Sp}(1)\times\mathrm{Sp}(1)$ *given by left and right multiplication by* $\mathbb{H}^*$ *is irreducible on* $\mathrm{Harm}_k(\mathbb{H},\mathbb{C})$*, i.e., for all nonzero* $f\in\mathrm{Harm}_k(\mathbb{H},\mathbb{C})$*, we have*

$$\mathrm{span}_{\mathbb{C}}\{q\mapsto f(q_1 q\overline{q_2}) : q_1,q_2\in\mathbb{H}^*\} = \mathrm{Harm}_k(\mathbb{H},\mathbb{C}). \tag{8.6}$$

We consider the special case of the linear polynomials ($k = 1$).

Example 8.6. The linear polynomial

$$f(q) := q = x_1 + ix_2 + jx_3 + kx_4, \qquad x_1, x_2, x_3, x_4 \in \mathbb{R},$$

is in $\mathrm{Harm}_1(\mathbb{H}, \mathbb{H}) = \mathrm{Hom}_1(\mathbb{H}, \mathbb{H})$, as are the coordinate maps $q \mapsto x_\ell$, which are also in $\mathrm{Harm}_1(\mathbb{H}, \mathbb{R}) = \mathrm{Hom}_1(\mathbb{H}, \mathbb{R})$. These can be written explicitly in the form (8.6) as follows

$$x_1 = \frac{1}{4}(q - iqi - jqj - kqk), \qquad x_2 = \frac{1}{4i}(q - iqi + jqj + kqk),$$
$$x_3 = \frac{1}{4j}(q + iqi - jqj + kqk), \qquad x_4 = \frac{1}{4k}(q + iqi + jqj - kqk). \tag{8.7}$$

The formula (8.7) is used by [24] to show that the "polynomials of degree k in q", i.e., sums of the "monomials"

$$q \mapsto a_0 q a_1 q a_2 \cdots q a_{k-1} q a_k, \qquad a_0, a_1, \ldots, a_k \in \mathbb{H},$$

is precisely $\mathrm{Hom}_k(\mathbb{H}, \mathbb{H})$ as we have defined it, or, equivalently, the $\mathbb{H}$-linear combinations of the monomials in real variables x_1, x_2, x_3, x_4.

9. The irreducible representations of $\mathrm{Harm}_k(\mathbb{H}^d, \mathbb{C})$

We now consider the irreducible representations of $\mathrm{Harm}_k(\mathbb{H}^d, \mathbb{C})$ for $d \geq 2$ (the case usually considered in the literature). Here, there is more than just the one irreducible W_k involved. Our method is to construct rectangular arrays, like that in Schematic 7.6, corresponding to a given irreducible $W_k, W_{k-2}, W_{k-4}, \ldots$. We will say that these are **commuting arrays** if we can move over them using L, R, L^*, R^*, as in the $d = 1$ case. They can be visualised as the "layers on (square) wedding cake".

We follow the development of Bachoc and Nebe [6]. For the action given by right multiplication by $\mathbb{H}^*$, let $I(W_p)^{(k)}$ be the homogeneous component of $\mathrm{Harm}_k(\mathbb{H}^d, \mathbb{C})$ corresponding to the irreducible W_p (of dimension $p + 1$), which gives the orthogonal decomposition

$$\mathrm{Harm}_k(\mathbb{H}^d, \mathbb{C}) = \bigoplus_{p \geq 0} I(W_p)^{(k)}.$$

The values of p involved in this sum are $p = k - 2j$, $0 \leq j \leq \frac{k}{2}$, which is observed in [6], and follows from our explicit decomposition (Theorem 9.1).

There is also the well known decomposition [17], [22] (Chapter 12, §12.2) into irreducibles for the action of left multiplication by $U(\mathbb{C}^{2d})$

$$\mathrm{Harm}_k(\mathbb{H}^d, \mathbb{C}) = \bigoplus_{a+b=k} H(a, b).$$

Since $U(\mathbb{H}^d)$ is the subgroup of $U(\mathbb{C}^{2d}) \subset \mathrm{O}(\mathbb{R}^{4d})$ characterised as those elements of $U(\mathbb{C}^{2d})$ which commute with right multiplication by $\mathbb{H}^*$ (in the group $\mathrm{O}(\mathbb{R}^{4d})$), we have the orthogonal direct sum of invariant $U(\mathbb{H}^d)$-modules

$$\mathrm{Harm}_k(\mathbb{H}^d, \mathbb{C}) = \bigoplus_{0 \leq j \leq \frac{k}{2}} \bigoplus_{j \leq b \leq k-j} H(k - b, b) \cap I(W_{k-2j})^{(k)}. \tag{9.1}$$

This is in fact an orthogonal direct sum of $U(\mathbb{H}^d)$-irreducibles

$$R_{k-2j}^{(k)} \cong H(k-b,b) \cap I(W_{k-2j})^{(k)}, \tag{9.2}$$

(see [16] §1.2, [6] Theorem 4.1, for k even).

We now give the irreducibles for right multiplication by $\mathbb{H}^*$. For $d = 1$, these were obtained by taking a row of the square array (7.11), i.e., by choosing a (particular) nonzero element $f \in H(k,0) = H(k,0) \cap \ker R^*$, and applying R to it k times. Since R^* moves back in the opposite direction to R, it followed that

$$\mathrm{span}_{\mathbb{C}}\{f, Rf, R^2 f, \ldots, R^k f\} \cong W_k \tag{9.3}$$

was an irreducible. Exactly the same argument holds for $d \geq 2$, i.e., for a nonzero $f \in H(k,0) = H(k,0) \cap \ker R^*$ the subspace (9.3) is irreducible. These are all the irreducibles for W_k, and for $d = 1$ this is the end of the story (Theorem 8.3). For $d \geq 2$, there are other irreducibles, constructed in a similar way: starting with a nonzero f in the second column, which does not give the irreducible W_k, i.e., $f \in H(k-1,1) \cap \ker R^*$, one obtains the irreducible subspaces

$$\mathrm{span}_{\mathbb{C}}\{f, Rf, R^2 f, \ldots, R^{k-2} f\} \cong W_{k-2},$$

and so forth.

Theorem 9.1. *For the action on* $\mathrm{Harm}_k(\mathbb{H}^d, \mathbb{C})$ *given by right multiplication by* $\mathbb{H}^*$*, the homogeneous component corresponding to the irreducible* W_{k-2j}*,* $0 \leq j \leq \frac{k}{2}$*, is*

$$\begin{aligned} I(W_{k-2j})^{(k)} &= \sum_{f \in H(k-j,j) \cap \ker R^*} \mathrm{span}_{\mathbb{C}}\{f, Rf, \ldots, R^{k-2j} f\} \qquad \text{(sum of irreducibles)}, \\ &= \bigoplus_{j \leq b \leq k-j} R^{b-j}(H(k-j,j) \cap \ker R^*) \qquad \text{(orthogonal direct sum)}. \end{aligned}$$

Moreover, these are the only irreducibles that appear, i.e., we have

$$\mathrm{Harm}_k(\mathbb{H}^d, \mathbb{C}) = \bigoplus_{0 \leq j \leq \frac{k}{2}} I(W_{k-2j})^{(k)} \qquad \text{(orthogonal direct sum)},$$

where the summands above are all nonzero for $d \geq 2$*, and* $\mathrm{Harm}_k(\mathbb{H}, \mathbb{C}) = I(W_k)^{(k)}$*.*

Proof. From Lemma 6.5, we have the orthogonal direct sum decomposition

$$\mathrm{Harm}_k(\mathbb{H}^d, \mathbb{C}) = \bigoplus_{0 \leq j \leq \frac{k}{2}} \bigoplus_{j \leq b \leq k-j} R^{b-j}(H(k-j,j) \cap \ker R^*),$$

and so it suffices to show that every nontrivial irreducible subspace

$$V \subset \bigoplus_{0 \leq a \leq k-2j} R^a(H(k-j,j) \cap \ker R^*) = \bigoplus_{j \leq b \leq k-j} R^{b-j}(H(k-j,j) \cap \ker R^*)$$

has the form

$$V = \mathrm{span}_{\mathbb{C}}\{f, Rf, \ldots, R^{k-2j} f\}, \quad f \in H(k-j,j) \cap \ker R^*,$$

so that $V \cong W_{k-2j}$.

Choose a nonzero $g \in V$, and write

$$g = \sum_{0 \le a \le k-2j} R^a f_a, \qquad f_a \in H(k-j,j) \cap \ker R^*.$$

Let a^* be the largest value of a for which $f_a \neq 0$. Then, by (5.17) of Lemma 5.5,

$$f := (R^*)^{a^*} g = (R^*)^{a^*} R^{a^*} f_{a^*} = (-a^*)_{a^*} (2j-k)_{a^*} f_{a^*},$$

which is a nonzero scalar multiple of f_{a^*} (for $2j-k=0$, $a^*=0$), and

$$W = \operatorname{span}_{\mathbb{C}}\{f, Rf, \ldots, R^{k-2j} f\} \subset V, \quad R^{k-2j} f \neq 0.$$

Hence $\dim(V) \ge \dim(W) = k-2j+1$. By construction, W is invariant under the action of R and R^*, and hence under right multiplication by $\mathbb{H}^*$ (Lemma 3.8). $\square$

We will call a sequence

$$f, Rf, \ldots, R^{k-2j} f, \qquad 0 \le j \le \frac{k}{2},$$

or any nonzero scalar multiples of it, an ***R*-orbit** (for W_{k-2j}) if

$$f \in \operatorname{Hom}_H(k-j,j), \qquad R^* f = 0.$$

It follows from Theorem 9.1 that $R^{k-2j} f \neq 0$, and

$$R\{f\} := \operatorname{span}_{\mathbb{C}}\{f, Rf, \ldots, R^{k-2j} f\}, \tag{9.4}$$

is an irreducible subspace (of dimension $k+1-2j$) for right multiplication by $\mathbb{H}^*$.

We can now give an explicit form for the $U(\mathbb{H}^d)$-irreducibles.

Theorem 9.2. *Let $d \ge 2$. For the action on $\operatorname{Harm}_k(\mathbb{H}^d, \mathbb{C})$ given by $U(\mathbb{H}^d) = \operatorname{Sp}(d)$, we have the following orthogonal direct sum of irreducibles*

$$\begin{aligned} \operatorname{Harm}_k(\mathbb{H}^d, \mathbb{C}) &= \bigoplus_{0 \le j \le \frac{k}{2}} \bigoplus_{j \le b \le k-j} H(k-b,b)_{k-2j} \\ &\cong \bigoplus_{0 \le j \le \frac{k}{2}} (k-2j+1) R^{(k)}_{k-2j}, \end{aligned} \tag{9.5}$$

where

$$\begin{aligned} H(k-b,b)_{k-2j} &:= R^{b-j}\big(\ker R^* \cap H(k-j,j)\big) \\ &= (R^*)^{k-b-j}\big(\ker R \cap H(j,k-j)\big) \\ &= H(k-b,b) \cap I(W_{k-2j})^{(k)} \\ &\cong R^{(k)}_{k-2j} := \ker R^* \cap H(k-j,j), \end{aligned} \tag{9.6}$$

and

$$\dim(R^{(k)}_{k-2j}) = (k-2j+1)(k+2d-1) \frac{(k-j+2d-2)!(j+2d-3)!}{(k-j+1)!j!(2d-1)!(2d-3)!}. \tag{9.7}$$

Proof. By Lemma 6.5, we already have that (9.5) is an orthogonal direct sum of $U(\mathbb{H}^d)$-invariant subspaces, with $H(k-b,b)_{k-2j} \subset H(k-b,b)$, given by the first two formulas in (9.6). We therefore need only show that they are $U(\mathbb{H}^d)$-irreducible, i.e., given by the formula (9.1), i.e., the third formula, with (9.2) holding (the fourth formula). By Theorem 9.1, we have

$$I(W_{k-2j})^{(k)} = \bigoplus_{j\le a\le k-j} R^a(H(k-j,j)\cap \ker R^*).$$

Since $R^a(H(k-j,j)\cap\ker R^*) \subset H(k-j-a,j+a)$, the only contribution to the intersection with $H(k-b,b)$ is when $b=j+a$, which gives the third formula, i.e.,

$$H(k-b,b)\cap I(W_{k-2j})^{(k)} = R^{b-j}\big(\ker R^*\cap H(k-j,j)\big).$$

We now show, that for j fixed, the $H(k-b,b)_{k-2j}$ are isomorphic $U(\mathbb{H}^d)$-irreducibles. Taking $\alpha=\beta=b-j$ in Lemma 6.1 gives

$$(R^*)^{b-j}H(k-b,b)_{k-2j} = (R^*)^{b-j}R^{b-j}\big(\ker R^*\cap H(k-j,j)\big)= \ker R^*\cap H(k-j,j).$$

This implies the subspaces have the same dimension as $\ker R^*\cap H(k-j,j)$, which is given by equation (6.18) of Lemma 6.6. Finally, since the action of $U(\mathbb{H}^d)$ commutes with the action of R (and its powers), these subspaces are all $U(\mathbb{H}^d)$-isomorphic to $R^{(k)}_{k-2j} := \ker R^*\cap H(k-j,j)$. □

This decomposition is given in [6] Theorem 4.1 (for k even, the summands not given explicitly), and in [4] (Theorems 1 and 2). The presentation of [4] involves two separate cases for the decomposition of $H(a,b)$, namely

$$H(k-b,b)_{k-2j} = \begin{cases} R^{b-j}(H(k-j,j)\cap\ker R^*), & k-b\ge b;\\ (R^*)^{k-b-j}(H(j,k-j)\cap \ker R), & k-b\le b.\end{cases}$$

We now consider the irreducibles for the action on $\mathrm{Harm}_k(\mathbb{H}^d,\mathbb{C})$ given by both left multiplication by $U\in U(\mathbb{H}^d)$ and right multiplication by $q^*\in\mathbb{H}^*$, i.e.,

$$((U,q^*)\cdot f)(q) := f(Uq\overline{q^*}).$$

Theorem 9.3. *For the action on* $\mathrm{Harm}_k(\mathbb{H}^d,\mathbb{C})$ *given by* $U(\mathbb{H}^d)\times\mathbb{H}^*$, $d\ge 2$, *we have the following orthogonal direct sum of irreducibles*

$$\mathrm{Harm}_k(\mathbb{H}^d,\mathbb{C}) = \bigoplus_{0\le j\le \frac{k}{2}} Q^{(k)}_{k-2j}, \tag{9.8}$$

where

$$\begin{aligned} Q^{(k)}_{k-2j} &:= \bigoplus_{j\le b\le k-j} H(k-b,b)_{k-2j}\\ &= \bigoplus_{j\le b\le k-j} R^{b-j}\big(\ker R^*\cap H(k-j,j)\big)= I(W_{k-2j})^{(k)}, \end{aligned}\tag{9.9}$$

and

$$\dim(Q^{(k)}_{k-2j}) = (k-2j+1)^2(k+2d-1)\frac{(k-j+2d-2)!(j+2d-3)!}{(k-j+1)!j!(2d-1)!(2d-3)!}. \tag{9.10}$$

Proof. The subspace $Q^{(k)}_{k-2j}$ is invariant under the actions of $U(\mathbb{H}^d)$ and $\mathbb{H}^*$, as it is a sum of irreducibles for each of these actions. We now show that it is irreducible.

Suppose $V \subset Q^{(k)}_{k-2j}$ is irreducible under the action of $U(\mathbb{H}^d) \times \mathbb{H}^*$. By Theorem 9.1, $V \subset I(W_{k-2j})^{(k)}$, and V contains an irreducible for the action of $\mathbb{H}^*$ of the form

$$\operatorname{span}_{\mathbb{C}}\{f, Rf, \ldots, R^{k-2j} f\}, \qquad 0 \neq R^{b-j} f \in H(k-b,b)_{k-2j}, \quad j \leq b \leq k-j.$$

Since each $H(k-b,b)_{k-2j}$ is $U(\mathbb{H}^d)$-irreducible, we have that $H(k-b,b)_{k-2j} \subset V$, and hence $V = Q^{(k)}_{k-2j}$ is irreducible. □

In other words, the $\mathrm{Sp}(d)\times\mathrm{Sp}(1)$-irreducibles $Q^{(k)}_{k-2j}$ are precisely the homogeneous components $I(W_{k-2j})$ for right multiplication by $\mathbb{H}^*$.

The decomposition (9.8) of $\mathrm{Harm}_k(\mathbb{H}^d,\mathbb{C})$ into $\mathrm{Sp}(d) \times \mathrm{Sp}(1)$-irreducibles is given in [23] Theorem 2.4, and [3] Proposition 2.1 (as the joint eigenfunctions of operators $\Delta_{\mathbb{S}}$ and Γ), where the following notations are used (respectively)

$$Q^{(k)}_{k-2j} = \begin{cases} H_{j,\frac{k}{2}-j}, & (k \text{ even}); \\ \tilde{H}_{j,\frac{k-1}{2}-j}, & (k \text{ odd}), \end{cases} \qquad\qquad Q^{(k)}_{k-2j} = \mathcal{H}_{k,j}.$$

Both observe that $Q^{(k)}_{k-2j}$ is invariant under conjugation, and so has a basis of real-valued polynomials, and a real-valued zonal function (a function invariant under the subgroup of $\mathrm{Sp}(d) \times \mathrm{Sp}(1)$ that fixes a point). The structural form of this zonal is given in [23] Proposition 2.8, and it is given explicitly in [3] Proposition 3.1.

The invariance of $Q^{(k)}_{k-2j}$ under conjugation follows directly from (4.7), i.e.,

$$\begin{aligned} \overline{H(k-b,b)_{k-2j}} &= \overline{R^{b-j}\big(\ker R^* \cap \mathrm{Hom}_H(k-j,j)\big)} \\ &= (R^*)^{b-j}\big(\overline{\ker R^*} \cap \overline{\mathrm{Hom}_H(k-j,j)}\big) \\ &= (R^*)^{b-j}\big(\ker R \cap \mathrm{Hom}_H(j,k-j)\big) \\ &= H(b,k-b)_{k-2j}. \end{aligned} \tag{9.11}$$

Schematic 9.4. *(Wedding cake) The orthogonal decomposition (9.5) of* $\mathrm{Harm}_k(\mathbb{H}^d,\mathbb{C})$ *into* $U(\mathbb{H}^d)$*-irreducibles can be displayed as layers of a "wedding cake"*

	$H(k,0)$	$H(k-1,1)$	$H(k-2,2)$	$\cdots$	$H(2,k-2)$	$H(1,k-1)$	$H(0,k)$
		$\overset{R^*}{\leftarrow}$		$\vdots$		$\overset{R}{\rightarrow}$	
$R^{(k)}_{k-4}$:			$H(k-2,2)_{k-4}$	$\cdots$	$H(2,k-2)_{k-4}$		
$R^{(k)}_{k-2}$:		$H(k-1,1)_{k-2}$	$H(k-2,2)_{k-2}$	$\cdots$	$H(2,k-2)_{k-2}$	$H(1,k-1)_{k-2}$	
$R^{(k)}_{k}$:	$H(k,0)_k$	$H(k-1,1)_k$	$H(k-2,2)_k$	$\cdots$	$H(2,k-2)_k$	$H(1,k-1)_k$	$H(0,k)_k$

where the layers (rows) correspond to the irreducible $R^{(k)}_{k-2j}$ *(the bottom layer is* $R^{(k)}_k$*), and the slices (columns) correspond to the decomposition of a given* $H(k-b,b)$ *into* $\min\{b,k-b\}+1$ *different irreducibles. One can move along the layers using* R *and* R^**, as indicated. Therefore, the left most irreducibles (shaded in grey), i.e.,*

$$H(k-j,j)_k = H(k-j,j) \cap \ker R^*, \qquad 0 \leq j \leq \frac{k}{2},$$

are a distinguished copy of each irreducible, from which the other summands in the layer can be obtained by applying R*. Further, in view of the symmetries (9.11), i.e., that conjugation reflects the cake around its* *centre, only half of these summands need be calculated, in practice. Similarly, the right most entries are distinguished, and give the other summands by applying R^*.*

Example 9.5. We consider $\mathrm{Harm}_2(\mathbb{H}^d,\mathbb{C})$, for which (2.5) gives

$$\dim(\mathrm{Harm}_2(\mathbb{H}^d,\mathbb{C})) = 2d(4d+1)-1 = (2d+1)(4d-1).$$

For $d=2$, we have the following table, where each line is an R-orbit, as in (9.4).

	$H(2,0)$	$H(1,1)$	$H(0,2)$
$K(2,0)$	z_1^2	$z_1\overline{w_1}$	$\overline{w_1}^2$
	z_2^2	$z_2\overline{w_2}$	$\overline{w_2}^2$
	z_1z_2	$z_1\overline{w_2}+z_2\overline{w_1}$	$\overline{w_1w_2}$
		$z_1\overline{w_2}-z_2\overline{w_1}$	
$K(1,1)$	z_1w_1	$z_1\overline{z_1}-w_1\overline{w_1}$	$\overline{z_1w_1}$
	z_1w_2	$\overline{w_1}w_2-z_1\overline{z_2}$	$\overline{w_1z_2}$
	z_2w_1	$\overline{w_2}w_1-z_2\overline{z_1}$	$\overline{w_2z_1}$
	z_2w_2	$z_2\overline{z_2}-w_2\overline{w_2}$	$\overline{z_2w_2}$
		$\overline{z_1}z_2+w_1\overline{w_2}$	
		$z_1\overline{z_2}+\overline{w_1}w_2$	
		$z_2\overline{z_2}+w_2\overline{w_2}-z_1\overline{z_1}-w_1\overline{w_1}$	
$K(0,2)$	w_1^2	$\overline{z_1}w_1$	$\overline{z_1}^2$
	w_2^2	$\overline{z_2}w_2$	$\overline{z_2}^2$
	w_1w_2	$\overline{z_1}w_2+\overline{z_2}w_1$	$\overline{z_1z_2}$
		$\overline{z_1}w_2-\overline{z_2}w_1$	

For example, we have the decomposition into irreducibles for right multiplication by $\mathbb{H}^*$

$$K(2,0) = (R\{z_1^2\}\oplus R\{z_2^2\}\oplus R\{z_1z_2\})\oplus R\{z_1\overline{w_2}-z_2\overline{w_1}\} \cong 3W_2\oplus W_0,$$

where

$$R\{z_1^2\} = \mathrm{span}\{z_1^2, z_1\overline{w_1}, \overline{w_1}^2\} \cong W_2, \quad R\{z_1\overline{w_2}-z_2\overline{w_1}\} = \mathrm{span}\{z_1\overline{w_2}-z_2\overline{w_1}\} \cong W_0,$$

etc. This calculation was done for $\mathrm{Hom}_2(\mathbb{H}^2)$, which has a dimension 1 higher. Apart from applying R to fill out the rows, the only other calculation done was solving $Rf=0$ or R^*f for $f\in H_2^{(1,1)}(\mathbb{H}^2) = K(1,1)\cap H(1,1)$ gives a 4-dimensional space spanned by

$$z_1\overline{z_1}+w_1\overline{w_1}, \quad z_2\overline{z_2}+w_2\overline{w_2}, \quad \overline{z_1}z_2+w_1\overline{w_2}, \quad z_1\overline{z_2}+\overline{w_1}w_2.$$

The first two have nonzero constant Laplacian, so their difference is harmonic, and the second two are harmonic. Similar calculations give the general decomposition

$$K(2,0) = \Big(\bigoplus_{|\alpha|=2} R\{z^\alpha\}\Big)\oplus\Big(\bigoplus_{1\le j<k\le d} R\{z_j\overline{w_k}-z_k\overline{w_j}\}\Big) \cong \frac{1}{2}d(d+1)W_2\oplus\frac{1}{2}d(d-1)W_0,$$

$$K(1,1) = \Big(\bigoplus_{1\le j,k\le d} R\{z_j w_k\}\Big) \oplus \Big(\bigoplus_{j\ne k} R\{z_j\overline{z_k} + \overline{w_j}w_k\} \oplus \bigoplus_{2\le j\le d} R\{z_j\overline{z_j} + w_j\overline{w_j} - z_1\overline{z_1} - w_1\overline{w_1}\}\Big)$$
$$\cong d^2 W_2 \oplus (d^2-1)W_0,$$
$$K(0,2) = \Big(\bigoplus_{|\alpha|=2} R\{w^\alpha\}\Big) \oplus \Big(\bigoplus_{1\le j<k\le d} R\{\overline{z_j}w_k - \overline{z_k}w_j\}\Big) \cong \frac{1}{2}d(d+1)W_2 \oplus \frac{1}{2}d(d-1)W_0,$$

into irreducibles (sums of R-orbits). In particular, the homogeneous components, i.e., the $\mathrm{Sp}(d)\times\mathrm{Sp}(1)$-irreducibles, are

$$\mathrm{Harm}_2(\mathbb{H}^d) = Q_2^{(k)} \oplus Q_0^{(k)} = I(W_2)^{(2)} \oplus I(W_0)^{(2)} \cong d(2d+1)W_3 \oplus (d-1)(2d+1)W_0.$$

Example 9.6. Since $H(k,0)\cap\ker R^* = H(k,0)$, we have

$$I(W_k)^{(k)} = \bigoplus_{|\alpha+\beta|=k} R\{z^\alpha w^\beta\} \cong \binom{k+2d-1}{k} W_k \quad \text{(orthogonal direct sum).}$$

10. Zonal polynomials

Here we consider the "zonal polynomials" for our irreducible representations of the groups $G = U(\mathbb{H}^d), U(\mathbb{H}^d)\times\mathbb{H}^*$ on $\mathrm{Harm}_k(\mathbb{H}^d,\mathbb{C})$. There are two common notions of zonal functions:

- The functions fixed by the action of the subgroup $G_{q'}$ which fixes a point q'.
- The Riesz representer of point evaluation at a point q' (the reproducing kernel).

When $G_{q'}$ is a maximal compact subgroup of G these are equivalent. We will consider the first notion. For a group G acting on $\mathbb{H}^d$, we define the stabiliser (or isotrophy) subgroup of $q'\in\mathbb{H}^d$ to be those elements which fix q', i.e.,

$$G_{q'} := \{g\in G : g\cdot q' = q'\}.$$

A function $\mathbb{H}^d\to\mathbb{C}$ which is fixed by the action of $G_{q'}$ is said to **zonal** (with pole q'). We denote the subspace of zonal functions in a space V of polynomials by

$$V^{G_{q'}} := \{f\in V : g\cdot f = f, \forall g\in G_{q'}\}.$$

We now consider the zonal polynomials for the group $G = U(\mathbb{H}^d)$.

Recall $\langle v,w\rangle = v^*w$ is the Euclidean inner product (2.2). For vectors $q = z + jw$, $q' = z' + jw'$ in $\mathbb{H}^d$, we define two inner products

$$\langle q',q\rangle_{\mathbb{H}^d} := \langle q',q\rangle \in \mathbb{H}, \qquad \langle q',q\rangle_{\mathbb{C}^{2d}} := \Big\langle \begin{pmatrix} z' \\ w' \end{pmatrix}, \begin{pmatrix} z \\ w \end{pmatrix} \Big\rangle \in \mathbb{C}. \tag{10.1}$$

Lemma 10.1. *For $q,q'\in\mathbb{H}^d$, we have*

$$\langle q',q\rangle_{\mathbb{H}^d} = \langle q',q\rangle_{\mathbb{C}^{2d}} + j\langle q'j,q\rangle_{\mathbb{C}^{2d}}. \tag{10.2}$$

For the action of $U(\mathbb{H}^d)$ the following are zonal polynomials $\mathbb{H}^d\to\mathbb{C}$ with pole q'

$$q \mapsto \langle q',q\rangle_{\mathbb{C}^{2d}} = \overline{z_1'}z_1 + \cdots + \overline{z_d'}z_d + \overline{w_1'}w_1 + \cdots + \overline{w_d'}w_d,$$

$$q \mapsto \langle q'j, q\rangle_{\mathbb{C}^{2d}} = z_1' w_1 + \cdots + z_d' w_d - w_1' z_1 - \cdots - w_d' z_d.$$

When $q' = e_1$, the zonal polynomials above are

$$z + jw \mapsto z_1, \qquad z + jw \mapsto w_1.$$

Proof. Using (2.1), we calculate

$$q'j = (z' + jw')j = z'j + jw'j = -\overline{w'} + j\overline{z'},$$

and so

$$\begin{aligned}\langle q', q\rangle_{\mathbb{H}^d} &= (z' + jw')^*(z + jw) = ((z')^* - (w')^* j)(z + jw) \\ &= (z')^* z + (w')^* w + j(\overline{z'})^* w - j(\overline{w'})^* z \\ &= \langle \begin{pmatrix} z' \\ w' \end{pmatrix}, \begin{pmatrix} z \\ w \end{pmatrix}\rangle + j\langle \begin{pmatrix} -\overline{w'} \\ \overline{z'} \end{pmatrix}, \begin{pmatrix} z \\ w \end{pmatrix}\rangle = \langle q', q\rangle_{\mathbb{C}^{2d}} + j\langle q'j, q\rangle_{\mathbb{C}^{2d}},\end{aligned}$$

which is (10.2). Let $U \in U(\mathbb{H}^d)$ with $Uq' = q'$, then we have

$$\langle q', q\rangle_{\mathbb{H}^d} = \langle Uq', Uq\rangle_{\mathbb{H}^d} = \langle q', Uq\rangle_{\mathbb{H}^d} = \langle q', Uq\rangle_{\mathbb{C}^{2d}} + j\langle q'j, Uq\rangle_{\mathbb{C}^{2d}},$$

so that

$$\langle q', Uq\rangle_{\mathbb{C}^{2d}} = \langle q', q\rangle_{\mathbb{C}^{2d}}, \qquad \langle q'j, Uq\rangle_{\mathbb{C}^{2d}} = \langle q'j, q\rangle_{\mathbb{C}^{2d}},$$

which shows that the linear polynomials given are zonal. □

We note that z_1 and w_1 are zonal polynomials in the $U(\mathbb{H}^d)$-irreducible subspace

$$H(1,0)_1 = \operatorname{span}\{z_1, \ldots, z_d, w_1, \ldots w_d\},$$

and so the space of zonal polynomials in a given $U(\mathbb{H}^d)$-irreducible is not 1-dimensional, as it is in the real and complex cases.

Example 10.2. The quadratic polynomial $q \mapsto \|q\|^2 = \langle q, q\rangle_{\mathbb{H}^d}$ is zonal (for any q'). By folk law (the real and complex cases), the zonal polynomials should be a function of this and the quaternionic inner product $q \mapsto \langle q', q\rangle_{\mathbb{H}^d} = (q')^* q$. Using (8.7), we have the explicit formulas:

$$\langle q', q\rangle_{\mathbb{C}^{2d}} = \frac{1}{2}(\langle q', q\rangle_{\mathbb{H}^d} - i\langle q', q\rangle_{\mathbb{H}^d} i), \qquad \langle q'j, q\rangle_{\mathbb{C}^{2d}} = \frac{1}{2j}(\langle q', q\rangle_{\mathbb{H}^d} + i\langle q', q\rangle_{\mathbb{H}^d} i).$$

Using the zonal polynomials above, which commute, since they are complex-valued, [6] define zonal polynomials in $\operatorname{Hom}_k(\mathbb{H}^d, \mathbb{C})$ by

$$[\alpha_1, \alpha_2, \alpha_3, \alpha_4, r]_{q'}(q) := \langle q', q\rangle_{\mathbb{C}^{2d}}^{\alpha_1} \langle q'j, q\rangle_{\mathbb{C}^{2d}}^{\alpha_2} \overline{\langle q', q\rangle}_{\mathbb{C}^{2d}}^{\alpha_3} \overline{\langle q'j, q\rangle}_{\mathbb{C}^{2d}}^{\alpha_4} \|q\|^{2r}, \tag{10.3}$$

where $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + 2r = k$. These span and hence are a basis for the zonal polynomials in $\operatorname{Hom}_k(\mathbb{H}^d, \mathbb{C})$ (Proposition 4.2, [6]).

Example 10.3. For a general q', we have

$$[\alpha_1,\alpha_2,\alpha_3,\alpha_4,r]_{q'} \in \mathrm{Hom}_H(\alpha_1+\alpha_2+r,\alpha_3+\alpha_4+r),$$

and for $q'=e_1$, we have

$$[\alpha_1,\alpha_2,\alpha_3,\alpha_4,r]_{e_1} = z_1^{\alpha_1}w_1^{\alpha_2}\overline{z_1}^{\alpha_3}\overline{w_1}^{\alpha_4}\|z+jw\|^{2r}, \quad \alpha_1+\alpha_2+\alpha_3+\alpha_4+2r=k, \tag{10.4}$$

so that

$$[\alpha_1,\alpha_2,\alpha_3,\alpha_4,r]_{e_1} \in \mathrm{Hom}_k^{(\alpha_2+\alpha_3+r,\alpha_3+\alpha_4+r)}(\mathbb{H}^d).$$

We can take advantage of (10.4) to simplify the proof and presentation of results, since if U is unitary with $Uq'=e_1$, then we have the following correspondence between zonal polynomials with poles q' and e_1

$$[a_1,a_2,a_3,a_4,r]_{q'} = [a_1,a_2,a_3,a_4,r]_{e_1}(V\cdot).$$

This follows from the calculation

$$\langle q',q\rangle_{\mathbb{H}^d} = \langle Uq',Uq\rangle_{\mathbb{H}^d} = \langle e_1,Uq\rangle_{\mathbb{H}^d},$$

and the fact that such a U can always be constructed, since $U(\mathbb{H}^d)$ is transitive on the quaternionic sphere. In effect, a zonal polynomial for q' can be obtained from one with pole e_1 by making the substitution

$$z_1 \mapsto \langle q',q\rangle_{\mathbb{C}^{2d}}, \qquad w_1 \mapsto \langle q'j,q\rangle_{\mathbb{C}^{2d}}. \tag{10.5}$$

We now show that R and L map the space of zonal polynomials for $q'=e_1$ to itself. This was assumed to be true for a general q', but calculations show otherwise (for L).

Lemma 10.4. *R and R^* map zonal polynomials to zonal polynomials, i.e.,*

$$R([a,b,c,d,r]) = a[a-1,b,c,d+1,r] - b[a,b-1,c+1,d,r], \tag{10.6}$$

$$R^*([a,b,c,d,r]) = -c[a,b+1,c-1,d,r] + d[a+1,b,c,d-1,r], \tag{10.7}$$

and L and L^* map zonal polynomials for $q'=z'+jw'$ as follows

$$\begin{aligned}L([a,b,c,d,r]) &= a[a-1,b,c,d,r][0,1,0,0,0]_{\overline{z'}} - d[a,b,c,d-1,r][0,0,1,0,0]_{\overline{z'}} \\ &\quad - b[a,b-1,c,d,r][1,0,0,0,0]_{\overline{jw'}} + c[a,b,c-1,d,r][0,0,0,1,0]_{\overline{jw'}},\end{aligned} \tag{10.8}$$

$$\begin{aligned}L^*([a,b,c,d,r]) &= b[a,b-1,c,d,r][1,0,0,0,0]_{\overline{z'}} - c[a,b,c-1,d,r][0,0,0,1,0]_{\overline{z'}} \\ &\quad - a[a-1,b,c,d,r][0,1,0,0,0]_{\overline{jw'}} + d[a,b,c,d-1,r][0,0,1,0,0]_{\overline{jw'}}.\end{aligned} \tag{10.9}$$

For $q'=z'\in\mathbb{R}^n$, L and L^* map zonal polynomials to zonal polynomials, i.e.,

$$L([a,b,c,d,r]) = a[a-1,b+1,c,d,r] - d[a,b,c+1,d-1,r], \tag{10.10}$$

$$L^*([a,b,c,d,r]) = b[a+1,b-1,c,d,r] - c[a,b,c-1,d+1,r]. \tag{10.11}$$

The action of Δ on zonal polynomials is similar to the real and complex cases.

Lemma 10.5. *The Laplacian maps zonal polynomials $\mathbb{H}^n \to \mathbb{C}$ to zonal polynomials, i.e.,*

$$\frac{1}{4}\Delta([a,b,c,d,r]) = ac[a-1,b,c-1,d,r] + bd[a,b-1,c,d-1,r] \\ + r(k+2n-1-r)[a,b,c,d,r-1]. \tag{10.12}$$

The number of zonal functions given by (10.3) is independent of the dimension d. For $d = 1$, these zonal polynomials have linear dependencies, e.g.,

$$[1,0,1,0,0] + [0,1,0,1,0] = z_1\overline{z_1} + w_1\overline{w_1} = [0,0,0,0,1],$$

and for $d > 1$ they are linearly dependent. Thus we obtain the following dimensions.

Lemma 10.6. *For $Z := U(\mathbb{H}^d)_{q'}$, $d \geq 2$, the zonal polynomials have dimensions*

$$\dim(\mathrm{Hom}_k(\mathbb{H}^d,\mathbb{C})^Z) = \sum_{0\leq j\leq \frac{k}{2}} \binom{k-2j+3}{3}, \tag{10.13}$$

$$\dim(\mathrm{Harm}_k(\mathbb{H}^d,\mathbb{C})^Z) = \binom{k+3}{3} = \sum_{0\leq j\leq \frac{k}{2}} (k-2j+1)^2. \tag{10.14}$$

Further, if $q' = z' \in \mathbb{C}^d$, e.g., $q' = e_1$, then

$$\dim(\mathrm{Hom}_k^{(a,b)}(\mathbb{H}^d)^Z) = \frac{1}{2}(m+1)(m+2), \tag{10.15}$$

$$\dim(H_k^{(a,b)}(\mathbb{H}^d)^Z) = m+1, \tag{10.16}$$

where

$$m := \min\{a, k-a, b, k-b\}.$$

Proof. Since the zonal polynomials in (10.4) are clearly linearly independent and span $\mathrm{Hom}_k(\mathbb{H}^d,\mathbb{C})^Z$ (see [6] Proposition 4.2), it suffices to count them, which gives

$$\dim(\mathrm{Hom}_k(\mathbb{H}^d,\mathbb{C})^Z) = \sum_{0\leq r\leq \frac{k}{2}} \dim(\mathrm{Hom}_{k-2r}(\mathbb{H},\mathbb{C})) = \sum_{0\leq r\leq \frac{k}{2}} \binom{k-2r+3}{3}.$$

When $q' = z'$ ($w' = 0$), each of these zonal polynomials is in some $\mathrm{Hom}_k^{(a,b)}(\mathbb{H}^d)$, so that

$$\mathrm{Hom}_k^{(a,b)}(\mathbb{H}^d)^Z = \bigoplus_{m\leq r\leq \frac{k}{2}} \mathrm{span}_{\mathbb{C}}\left\{[\alpha_1,\alpha_2,\alpha_3,\alpha_4,r] : \begin{matrix} \alpha_1+\alpha_4 = k-a-r, & \alpha_2+\alpha_3 = a-r \\ \alpha_1+\alpha_2 = k-b-r, & \alpha_3+\alpha_4 = b-r \end{matrix}\right\}, \tag{10.17}$$

and counting again, using (7.4) and $m_{a-r,b-r}^{(k-2r)} = m+1-r$, gives

$$\dim(\mathrm{Hom}_k^{(a,b)}(\mathbb{H}^d)^Z) = \sum_{m\leq r\leq \frac{k}{2}} \dim(\mathrm{Hom}_{k-2r}^{(a-r,b-r)}(\mathbb{H})) \\ = 1+2+\cdots+m+(m+1) = \frac{1}{2}(m+1)(m+2).$$

Since the Laplacian maps $\mathrm{Hom}_k^{(a,b)}(\mathbb{H}^d)$ onto $\mathrm{Hom}_{k-1}^{(a-1,b-1)}(\mathbb{H}^d)$ and zonal polynomials to zonal polynomials (Lemma 10.5), we have

$$\dim(\mathrm{Harm}_k(\mathbb{H}^d,\mathbb{C})^Z) = \dim(\mathrm{Hom}_k(\mathbb{H}^d,\mathbb{C})^Z) - \dim(\mathrm{Hom}_{k-2}(\mathbb{H}^d,\mathbb{C})^Z) = \binom{k+3}{3},$$
$$\begin{aligned}\dim(H_k^{(a,b)}(\mathbb{H}^d)^Z) &= \dim(\mathrm{Hom}_k^{(a,b)}(\mathbb{H}^d)^Z) - \dim(\mathrm{Hom}_{k-2}^{(a-1,b-1)}(\mathbb{H}^d)^Z)\\ &= \frac{1}{2}(m+1)(m+2) - \frac{1}{2}(m-1+1)(m-1+2) = m+1,\end{aligned}$$

which completes the proof. □

We will give a simple example first, which motivates the general and constructive result to follow.

Example 10.7. For $q' = e_1$, the unique zonal polynomial in $H_k^{(0,0)}(\mathbb{H}^d)$ is

$$[k,0,0,0,0] = z_1^k.$$

We may apply L (down) and R (right) to this, as in the univariate case (Schematic 7.6 and Lemma 8.2), to obtain $(k+1)^2$ zonal polynomials in $I(W_k)^{(k)}$.

$$\begin{array}{cccc}
z_1^k & kz_1^{k-1}\overline{w_1} & \cdots & k!\overline{w_1}^k\\
kz_1^{k-1}w_1 & k(k-1)z_1^{k-2}w_1\overline{w_1} - kz_1^{k-1}\overline{z_1} & \cdots & -k!k\overline{z_1}\,\overline{w_1}^{k-1}\\
k(k-1)z_1^{k-2}w_1^2 & k(k-1)\{(k-2)z_1^{k-3}w_1^2\overline{w_1} - 2z_1^{k-2}w_1\overline{z_1}\} & \cdots & k!k(k-1)\overline{z_1}^2\overline{w_1}^{k-2}\\
\vdots & \vdots & & \vdots\\
k!z_1w_1^{k-1} & k!w_1^{k-1}\overline{w_1} - k!(k-1)z_1w_1^{k-2}\overline{z_1} & \cdots & (-1)^{k-1}k!^2\overline{z_1}^{k-1}\overline{w_1}\\
k!w_1^k & -k!kw_1^{k-1}\overline{z_1} & \cdots & (-1)^k k!^2\overline{z_1}^k
\end{array}$$

Theorem 10.8. *Let $q' = e_1$. For $d \ge 2$, there is a unique harmonic zonal polynomial*

$$P_{k-2j}^{(k)} = P_{k-2j,d}^{(k)} \in \ker L^* \cap \ker R^* \cap H_k^{(j,j)}(\mathbb{H}^d), \qquad 0 \le j \le \frac{k}{2},$$

given by

$$P_{k-2j}^{(k)} := \sum_{b+c+r=j} \frac{(-1)^r}{b!c!r!} \frac{(k+2-j-r)_r}{(k+2d-1-r)_r}[k-j-b-r,b,c,b,r], \tag{10.18}$$

which has $\frac{1}{2}(j+1)(j+2)$ terms. Let

$$P_{k-2j,a,b}^{(k)} := L^{a-j}R^{b-j}P_{k-2j}^{(k)}, \qquad j \le a,b \le k-j. \tag{10.19}$$

Then the zonal polynomials (with pole e_1) in $\mathrm{Harm}_k(\mathbb{H}^d,\mathbb{C})$ have the following orthogonal direct sum decomposition into one-dimensional subspaces

$$\mathrm{Harm}_k(\mathbb{H}^d,\mathbb{C})^Z = \bigoplus_{0\le j\le \frac{k}{2}} \bigoplus_{j\le a,b\le k-j} \mathrm{span}\{P_{k-2j,a,b}^{(k)}\}. \tag{10.20}$$

Proof. By (10.17), a general zonal polynomial in $\mathrm{Hom}_k^{(j,j)}(\mathbb{H}^d)$ has the form

$$f := \sum_{b+c+r=j} C_{bcr}[k-j-b-r,b,c,b,r], \qquad C_{bcr} \in \mathbb{C},$$

which involves $\frac{1}{2}(j+1)(j+2)$ terms. By Lemma 10.5, the condition for f to be harmonic is

$$\frac{1}{4}\Delta f = \sum_{b+c+r=j} C_{bcr}\big\{(k-j-b-r)c[k-j-b-r-1,b,c-1,b,r] + b^2[k-j-b-r,b-1,c,b-1,r] + r(k+2d-1-r)[k-j-b-r,b,c,b,r-1]\big\} = 0,$$

which gives $\frac{1}{2}j(j+1)$ equations, and hence $j+1 = \dim((H_k^{(j,j)})^Z)$ free parameters. Hand calculations indicated that $\Delta f = 0$ together with the conditions $R^* f = 0$ and $L^* f = 0$ leads to a unique (one parameter) solution f. From these special cases, we "guessed" the formula (10.18). We will now verify directly that f defined by (10.18) has the desired properties, and then conclude that it is unique (by a cardinality argument).

By Lemma 10.5 and Lemma 10.4, we have

$$\begin{aligned}
\Delta([k-j-b-r,b,c,b,r]) &= (k-j-b-r)c[k-j-b-r-1,b,c-1,b,r] \\
&\quad + b^2[k-j-b-r,b-1,c,b-1,r] \\
&\quad + r(k+2d-1-r)[k-j-b-r,b,c,b,r-1], \\
R^*([k-j-b-r,b,c,b,r]) &= -c[k-j-b-r,b+1,c-1,b,r] \\
&\quad + b[k-j-b-r+1,b,c,b-1,r], \\
L^*([k-j-b-r,b,c,b,r]) &= b[k-j-b-r+1,b-1,c,b,r] \\
&\quad - c[k-j-b-r,b,c-1,b+1,r].
\end{aligned}$$

Hence, the $[k-j-b'-r'-1,b',c',b',r']$ coefficient of Δf is

$$\frac{(-1)^{r'}}{(b')!(c')!}\frac{(k+2-j-r')_{r'}}{(k+2d-1-r')_{r'}}\Big\{\frac{1}{c'+1}(k-j-b'-r')(c'+1) + \frac{1}{b'+1}(b'+1)^2 - \frac{1}{r'+1}\frac{(k+2-j-r'-1)}{(k+2d-1-r'-1)}(r'+1)(k+2d-1-r'-1)\Big\} = 0,$$

the $[k-j-b'-r+1,b',c',b'-1,r]$, $b'\neq 0$, coefficient of $R^* f$ is

$$\frac{(-1)^r}{r!}\frac{(k+2-j-r)_r}{(k+2d-1-r)_r}\Big(\frac{1}{(b'-1)!(c'+1)!}(-(c'+1)) + \frac{1}{(b')!(c')!}b'\Big) = 0,$$

and the $[k-j-b'-r+1,b'-1,c',b',r]$, $b'\neq 0$, coefficient of $L^* f$ is

$$\frac{(-1)^r}{r!}\frac{(k+2-j-r)_r}{(k+2d-1-r)_r}\Big\{\frac{1}{(b')!(c')!}(b') - \frac{1}{(b'-1)!(c'+1)!}(c'+1)\Big\} = 0,$$

so that $f = P_{k-2j}^{(k)} \in \ker(L^*)\cap\ker(R^*)\cap H_k^{(j,j)}(\mathbb{H}^d)^Z$. Since $P_{k-2j}^{(k)} \in H_k^{(j,j)}(\mathbb{H}^d)$, by Lemma 6.4 and Lemma 10.4, we have the orthogonal direct sum decomposition

$$\bigoplus_{0\le j\le \frac{k}{2}} \bigoplus_{j\le a,b\le k-j} \operatorname{span}\{L^{a-j}R^{b-j}P_{k-2j}^{(k)}\} \subset \operatorname{Harm}_k(\mathbb{H}^d,\mathbb{C})^Z,$$

and by a dimension count using (10.14), we obtain (10.20), and hence the uniqueness of $P_{k-2j}^{(k)}$ up to a scalar multiple. □

It follows from Theorem 10.8 (also see [6]) the zonal functions satisfy

$$\dim((I(W_{k-2j})^{(k)})^Z) = (k-2j+1)^2,$$
$$\dim(H(k-b,b)^Z_{k-2j}) = k-2j+1, \quad j \le b \le k-j,$$

and for $q' = e_1$, we have

$$\dim(H(k-b,b)_{k-2j} \cap H_k^{(a,b)}(\mathbb{H}^d)^Z) = \begin{cases} 1, & j \le a, b \le k-j; \\ 0, & \text{otherwise}. \end{cases} \tag{10.21}$$

Let $Z^{(k)}_{k-2j,a,b}$ be the zonal polynomial with pole q' obtained from $P^{(k)}_{k-2j,a,b}$ by making the substitution (10.5).

Corollary 10.9. *The zonal polynomials in* $\mathrm{Harm}_k(\mathbb{H}^d, \mathbb{C})$ *have the following orthogonal direct sum decomposition into one-dimensional subspaces*

$$\mathrm{Harm}_k(\mathbb{H}^d, \mathbb{C})^Z = \bigoplus_{0 \le j \le \frac{k}{2}} \bigoplus_{j \le a,b \le k-j} \mathrm{span}\{Z^{(k)}_{k-2j,a,b}\}. \tag{10.22}$$

Proof. Apply the substitution (10.5) to the orthogonal direct sum (10.20). □

The existence of the zonal polynomials $Z^{(k)}_{k-2j,a,b}$ in (10.20) is proved inductively in [6], where they are denoted by $\mathcal{Z}^{(k)}_{p,w,w'}$. We now outline how the two are related. Here $p = k-2j$, and the "weight" parameters w, w' are related to (a, b), as follows

$$a = \frac{k-w'}{2}, \quad b = \frac{k-w}{2}, \qquad w' = k-2a, \quad w = k-2b, \tag{10.23}$$

which gives the correspondence between indices

$$(a, b) \in \{0, 1, \dots, k\}^2 \iff (w, w') \in \{-k, -k+2, \dots, k-2, k\}^2.$$

We note that for k even (the case considered in [6]) the weights w and w' are even, and for k odd, they are odd. They define the space of zonal polynomials

$$\begin{aligned} E^{(k)}_{w,w'} := \mathrm{span}\{[\alpha_1, \alpha_2, \alpha_3, \alpha_4, r]_{q'} : \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + 2r = k, \\ \alpha_1 + \alpha_2 - \alpha_3 - \alpha_4 = w, \alpha_1 - \alpha_2 - \alpha_3 + \alpha_4 = w'\}, \end{aligned} \tag{10.24}$$

which satisfies

$$E^{(k)}_{w,w'} = \mathrm{Hom}^{(a,b)}_k(\mathbb{H}^d)^Z, \qquad \text{for } q' = z' \in \mathbb{C}^d,$$

and the space

$$V^{(k)}_w = H\Big(\frac{k+w}{2}, \frac{k-w}{2}\Big) = H(k-b, b).$$

In [6] (Proposition 4.5), the zonal polynomials $\mathcal{Z}^{(k)}_{p,w,w'}$ are characterised by the following properties:

- $\mathcal{Z}_{p,w,w'} \in E^{(k)}_{w,w'}$, i.e., $Z^{(k)}_{k-2j,a,b}$ has the structural form given by (10.18) and (10.19).

- $\{\mathcal{Z}_{p,w,w'}\}_{w'\in\{-p,\ldots,p-2,p\}}$ is a basis of (the zonal polynomials in) $I(W_p)^{(k)}\cap V_w^{(k)}$, i.e., $\{Z^{(k)}_{k-2j,a,b}\}_{j\le a\le k-j}$ is a basis of the zonal polynomials in $I(W_{k-2j})^{(k)}\cap H(k-b,b)$.
- $\{\mathcal{Z}_{p,w,w'}\}_{w\in\{-p,\ldots,p-2,p\}}$ is a basis of zonal polynomials for an irreducible subspace for right multiplication by $\mathbb{H}^*$, (which is isomorphic to W_p), i.e., $\{Z^{(k)}_{k-2j,a,b}\}_{j\le b\le k-j}$ is an R-orbit for a W_{k-2j}.

These follow from our construction, and the observation (by Lemma 10.4) that

$$Z^{(k)}_{k-2j,a,b}=R^{b-j}Z^{(k)}_{k-2j,a,0},\qquad j\le b\le k-j.$$

Example 10.10. The first three polynomials $Z^{(k)}_{k-2j}=Z^{(k)}_{k-2j,0,0}$ given by (10.18) are

$$\begin{aligned}
Z^{(k)}_k&=[k,0,0,0,0],\\
Z^{(k)}_{k-2}&=[k-2,1,0,1,0]+[k-1,0,1,0,0]-\frac{k}{k+2d-2}[k-2,0,0,0,1],\\
Z^{(k)}_{k-4}&=[k-4,2,0,2,0]+[k-2,0,2,0,0]+2[k-3,1,1,1,0]-\frac{2(k-1)}{k+2d-2}[k-4,1,0,1,1]\\
&\qquad-\frac{2(k-1)}{k+2d-2}[k-3,0,1,0,1]+\frac{(k-1)(k-2)}{(k+2d-2)(k+2d-3)}[k-4,0,0,0,2].
\end{aligned}$$

We observe that, except for the first, these depend on the dimension d.

Example 10.11. For $k=1$, the zonal polynomials in (10.22) are

$$\begin{aligned}
Z_1^{(1)}&:=[1,0,0,0,0]=z_1, & RZ_1^{(1)}&=[0,0,0,1,0]=\overline{w_1},\\
LZ_1^{(1)}&=[0,1,0,0,0]=w_1, & -LRZ_1^{(1)}&=[0,0,1,0,0]=\overline{z_1},
\end{aligned}$$

and for $k=2$, they are given by the schematic

	$H(1,1)_0$
$K(1,1)$	$[1,0,1,0,0]+[0,1,0,1,0]-\frac{1}{d}[0,0,0,0,1]$

	$H(2,0)_2$	$H(1,1)_2$	$H(0,2)_2$
$K(2,0)$	$[2,0,0,0,0]$	$[1,0,0,1,0]$	$[0,0,0,2,0]$
$K(1,1)$	$[1,1,0,0,0]$	$[0,1,0,1,0]-[1,0,1,0,0]$	$[0,0,1,1,0]$
$K(0,2)$	$[0,2,0,0,0]$	$[0,1,1,0,0]$	$[0,0,2,0,0]$

with the indexing of rows and columns as before.

Similarly to the Schematic 9.4, the summands $\{Z_{k-2j,a,b}\}_{j\le a,b\le k-j}$, $0\le j\le\frac{k}{2}$, in (10.22) can be arranged as the layers of a "wedding cake" (see Fig. 1).

We now seek an explicit formula for the zonal polynomial $L^\alpha R^\beta P^{(k)}_{k-2j}$ of Theorem 10.8. We first determine its structural form. The Lemma 10.12, below, says that the complexity of the formula depends on how far the index (α,β) is from the edges of the array of indices $A=\{0,1,\ldots,k-2j\}^2$. Partition A into "nested squares", as in (8.4),

$$S_{k,j,m}:=\{(\alpha,\beta):\min\{j+\alpha,j+\beta,k-j-\alpha,k-j-\beta\}=m\},\quad j\le m\le\frac{k}{2}. \tag{10.25}$$

Lemma 10.12. *Let $0\le\alpha,\beta\le k-2j$, $0\le j\le\frac{k}{2}$, and*

$$m:=\min\{j+\alpha,k-j-\beta,j+\beta,k-j-\alpha\}\qquad \textit{i.e., } (\alpha,\beta)\in S_{k,j,m}.$$

			$H(2,2)_0$		
$K(2,2)$			$P_0 = P^{(4)}_{000}$		
		$H(3,1)_2$	$H(2,2)_2$	$H(1,3)_2$	
$K(3,1)$		$P_2 = P^{(4)}_{200}$	RP_2	R^2P_2	
$K(2,2)$		LP_2	LRP_2	LR^2P_2	
$K(1,3)$		L^2P_2	L^2RP_2	$L^2R^2P_2$	
	$H(4,0)_4$	$H(3,1)_4$	$H(2,2)_4$	$H(1,3)_4$	$H(0,4)_4$
$K(4,0)$	$P_4 = P^{(4)}_{400}$	RP_4	R^2P_4	R^3P_4	R^4P_4
$K(3,1)$	LP_4	LRP_4	LR^2P_4	LR^3P_4	LR^4P_4
$K(2,2)$	L^2P_4	L^2RP_4	$L^2R^2P_4$	$L^2R^3P_4$	$L^2R^4P_4$
$K(1,3)$	L^3P_4	L^3RP_4	$L^3R^2P_4$	$L^3R^3P_4$	$L^3R^4P_4$
$K(0,4)$	L^4P_4	L^4RP_4	$L^4R^2P_4$	$L^4R^3P_4$	$L^4R^4P_4$

Fig. 1. Schematic of the $1^2 + 3^2 + 5^2$ zonal functions for $\mathrm{Harm}_4(\mathbb{H}^d, \mathbb{C})$ given by (10.19).

Then $L^\alpha R^\beta P^{(k)}_{k-2j} \in K(k-j-\alpha, j+\alpha) \cap H(k-j-\beta, j+\beta)$ *has the form*

$$L^\alpha R^\beta P^{(k)}_{k-2j} = \sum_{\substack{0\le r\le j\\ 0\le b\le m-r}} C^{(\alpha,\beta)}_{br}[k-j-\beta-b-r, b, \alpha+j-r-b, \beta-\alpha+b, r], \quad \alpha \le \beta,$$

$$L^\alpha R^\beta P^{(k)}_{k-2j} = \sum_{\substack{0\le r\le j\\ 0\le b\le m-r}} C^{(\alpha,\beta)}_{br}\text{‘}[k-j-\alpha-b-r, \alpha-\beta+b, j+\beta-b-r, b, r], \quad \alpha \ge \beta,$$

which involves $\frac{1}{2}(j+1)(2m+2-j)$ *terms.*

Proof. A general zonal polynomial of degree k has the form

$$f = \sum_{a+b+c+d+2r=k} C_{abcdr}[a,b,c,d,r].$$

By Lemma 10.4, L and R applied to $[a,b,c,d,r]$ preserves the value of r, so that

$$f = L^\alpha R^\beta P^{(k)}_{k-2j}$$

has the same restriction on r as $P^{(k)}_{k-2j}$ does, i.e., $0 \le r \le j$.

The condition that $f = L^\alpha R^\beta P^{(k)}_{k-2j} \in K(k-j-\alpha, j+\alpha) \cap H(k-j-\beta, j+\beta)$ gives

$$\begin{aligned} a+b+r &= k-j-\beta, \qquad & c+d+r &= j+\beta, \\ a+d+r &= k-j-\alpha, \qquad & b+c+r &= j+\alpha. \end{aligned} \tag{10.26}$$

First consider the case $\alpha \le \beta$, i.e., $m = j+\alpha$ or $m = k-j-\beta$. If $m = j+\alpha$, then (10.26) gives

$$b+c = m-r, \qquad a = k-j-\beta-b-r, \qquad d = j+\beta-c-r,$$

so that

$$L^\alpha R^\beta P^{(k)}_{k-2j} = \sum_{\substack{0\le r\le j\\ b+c=m-r}} C_{bcr}[k-j-\beta-b-r, b, c, j+\beta-c-r, r], \quad m = j+\alpha.$$

Using $b+c+r = j+\alpha$ to eliminate c above, gives

$$L^\alpha R^\beta P^{(k)}_{k-2j} = \sum_{\substack{0\le r\le j\\ 0\le b\le m-r}} C_{br}[k-j-\beta-b-r, b, j+\alpha-r-b, \beta-\alpha+b, r]. \tag{10.27}$$

Now consider $m = k - j - \beta$. Then (10.26) gives

$$a + b = m - r, \quad a = k - j - \beta - b - r, \quad c = j + \alpha - r - b, \qquad d = \beta - \alpha + b,$$

so that (10.27) holds for $m = j + \alpha$ and $m = k - j - \beta$, i.e., $\alpha \leq \beta$.

For the case $\alpha \geq \beta$, i.e., $m = j + \beta$ or $m = k - j - \alpha$, we have, respectively

$$c + d = m - r, \quad a = k - j - \alpha - d - r, \quad b = \alpha - \beta + d, \quad c = j + \beta - d - r,$$
$$a + d = m - r, \quad a = k - j - \alpha - d - r, \quad b = \alpha - \beta + d, \quad c = j + \beta - d - r,$$

which (replacing d by b) gives the second formula.

In the sum, we can have $r = 0, 1, \ldots, j$ ($j + 1$ choices), with $m + 1 - r$ choices for b, and so the number of terms is

$$(m + 1) + m + (m - 1) + \cdots + (m + 1 - j) = \frac{1}{2}(j + 1)(2m + 2 - j). \quad \square$$

An explicit formula for $L^\alpha R^\beta P_{k-2j}^{(k)}$ can by found by applying (10.6) and (10.10). This gives very complicated coefficients. Instead, we used Lemma 8.2 and numerous symbolic calculations for low values of α and β, such as Lemma 10.13 below, to conjecture the formulas of Theorems 10.14 and 10.16, which were then proved for a general (α, β).

Lemma 10.13. *For $0 \leq \beta \leq k - 2j$, $0 \leq j \leq \frac{k}{2}$, we have*

$$R^\beta Z_{k-2j}^{(k)} = \sum_{b+c+r=j} \frac{(-1)^r}{b!c!r!} \frac{(k + 2 - j - r)_r}{(k + 2d - 1 - r)_r} (k - 2j - \beta + 1)_\beta [k - 2j - \beta + c, b, c, b + \beta, r].$$

Proof. Use induction on $0 \leq \beta \leq k - 2j$, see [21] for details. $\square$

Theorem 10.14. *For $0 \leq \alpha \leq \beta \leq k - 2j$, $0 \leq j \leq \frac{k}{2}$, we have*

$$L^\alpha R^\beta Z_{k-2j}^{(k)} = \sum_{\substack{0 \leq r \leq j \\ 0 \leq b \leq m - r}} C_{br}^{(\alpha,\beta)} [k - j - \beta - b - r, b, j + \alpha - r - b, \beta - \alpha + b, r],$$

where $m = \min\{j + \alpha, k - j - \beta\}$ and $C_{br}^{(\alpha,\beta)} = A_{br}^{(\alpha,\beta)} B_{br}^{(\alpha,\beta)}$, with

$$A_{br}^{(\alpha,\beta)} := \frac{(-1)^r}{b!(j + \alpha - b - r)!r!} \frac{(k + 2 - j - r)_r}{(k + 2d - 1 - r)_r} (k - 2j - \beta + 1)_\beta, \tag{10.28}$$

$$\begin{aligned} B_{br}^{(\alpha,\beta)} &:= \sum_{u+v=\alpha} \frac{\alpha!}{u!v!} (k - 2j - \alpha + 1)_u (b - u + 1)_u (j - r + 1)_v (-\beta)_v \\ &= (1 + j - r)_\alpha (-\beta)_\alpha \, {}_3F_2\left(\begin{matrix} -\alpha, -b, k - 2j - \alpha + 1 \\ r - j - \alpha, \beta + 1 - \alpha \end{matrix} ; 1 \right). \end{aligned} \tag{10.29}$$

The constant $B_{br}^{(\alpha,\beta)}$ can also be calculated from $B_{br}^{(0,\beta)} := 1$ and the recurrence

$$B_{br}^{(\alpha,\beta)} = (k - j - \beta - b + 1 - r) b B_{b-1,r}^{(\alpha-1,\beta)} - (\beta - \alpha + 1 + b)(j + \alpha - b - r) B_{br}^{(\alpha-1,\beta)}. \tag{10.30}$$

Proof. We first prove the result for $B_{br}^{(\alpha,\beta)}$ given by the recurrence relation (10.30), by using induction on α. This is true for $\alpha = 0$ and all β by Lemma 10.13 (where $m = j$). Let $A_{-1,r}^{(\alpha,\beta)}$ and $B_{-1,r}^{(\alpha,\beta)}$ take some value (it matters not which). Then we have

$$A_{b-1,r}^{(\alpha-1,\beta)} = bA_{br}^{(\alpha,\beta)}, \quad A_{br}^{(\alpha-1,\beta)} = (j+\alpha-b-r)A_{br}^{(\alpha,\beta)}, \qquad \alpha > 0. \tag{10.31}$$

Suppose that $\alpha > 0$, then by the inductive hypothesis, we have

$$L^{\alpha-1}R^\beta Z_{k-2j}^{(k)} = \sum_{\substack{0\le r\le j \\ 0\le b'\le m-r}} C_{b'r}^{(\alpha-1,\beta)}[k-j-\beta-b'-r, b', j+\alpha-1-r-b', \beta-\alpha+1+b', r].$$

We apply L to this, using (10.10), i.e.,

$$\begin{aligned}
&L([k-j-\beta-b'-r, b', j+\alpha-1-r-b', \beta-\alpha+1+b', r]) \\
&\quad = (k-j-\beta-b'-r)[k-j-\beta-b'-r-1, b'+1, j+\alpha-1-r-b', \beta-\alpha+1+b', r] \\
&\qquad - (\beta-\alpha+1+b')[k-j-\beta-b'-r, b', j+\alpha-r-b', \beta-\alpha+b', r]
\end{aligned}$$

and (10.31), to obtain

$$\begin{aligned}
C_{br}^{(\alpha,\beta)} &= (k-j-\beta-(b-1)-r)C_{b-1,r}^{(\alpha-1,\beta)} - (\beta-\alpha+1+b)C_{br}^{(\alpha-1,\beta)} \\
&= (k-j-\beta-b+1-r)bA_{br}^{(\alpha,\beta)}B_{b-1,r}^{(\alpha,\beta)} - (\beta-\alpha+1+b)(j+\alpha-b-r)A_{br}^{(\alpha,\beta)}B_{br}^{(\alpha-1,\beta)}.
\end{aligned}$$

Since $A_{br}^{(\alpha,\beta)} \ne 0$, we may divide the above by it, to obtain

$$B_{br}^{(\alpha,\beta)} = (k-j-\beta-b+1-r)bB_{b-1,r}^{(\alpha-1,\beta)} - (\beta-\alpha+1+b)(j+\alpha-b-r)B_{br}^{(\alpha-1,\beta)},$$

i.e., (10.30), which completes the induction.

Finally, we show that the formula (10.29) for $B_{br}^{(\alpha,\beta)}$ involving a ${}_3F_2$ hypergeometric series holds, i.e., it satisfies the recurrence. This we do by induction on α. The case $\alpha = 0$ is immediate, and the inductive step follows from the contiguous relation

$$\begin{aligned}
(de)\,{}_3F_2\Big({-n, a, c \atop d, e}; 1\Big) &= (a+c-d-e+1-n)(-a)\,{}_3F_2\Big({1-n, a+1, c+1 \atop d+1, e+1}; 1\Big) \\
&\quad - (e-a)(a-d)\,{}_3F_2\Big({1-n, a, c+1 \atop d+1, e+1}; 1\Big),
\end{aligned}$$

for hypergeometric functions, for the choice

$$n = \alpha, \quad a = -b, \quad c = k-2j-\alpha+1, \quad d = r-j-\alpha, \quad e = \beta+1-\alpha. \quad \square$$

The recurrence relation (10.30) was determined first. It suggests that $(b,\beta) \mapsto B_{br}^{\alpha,\beta}$ is a polynomial of degree 2α, where in fact it has degree α, as is indicated by (10.29). We could not prove formula (10.29) directly, without recourse to the contiguous relation. To indicate the complexity of such a calculation, we give the inductive step for $\alpha = 1, 2$

$$\begin{aligned}
B_{br}^{(1,\beta)} &= (k-j-\beta-b+1-r)b - (\beta+b)(j+1-b-r) = (k-2j)b - (j-r+1)\beta, \\
B_{br}^{(2,\beta)} &= (k-j-\beta-b+1-r)b\{(k-2j)(b-1) - (j-r+1)\beta\}
\end{aligned}$$

$$\begin{aligned}&- (\beta-1+b)(j+2-b-r)\{(k-2j)b-(j-r+1)\beta\}\\&=(j-r+1)_2(-\beta)_2+2(k-2j-1)b(j-r+1)(-\beta)+(k-2j-2)_2(b-1)_2.\end{aligned}$$

Example 10.15. For $r=j$, the hypergeometric series in (10.29) can be summed using the generalised binomial theorem, or Gauss's summation for the resulting ${}_2F_1$, to obtain

$$\begin{aligned}B_{bj}^{(\alpha,\beta)}&=\sum_{u=0}^{b}\frac{\alpha!}{u!v!}(k-2j-\alpha+1)_u\frac{b!}{(b-u)!}v!(-\beta)_{\alpha-b}(-\beta+\alpha-b)_{b-u}\\&=\alpha!(-\beta)_\alpha\frac{(\beta-k+2j)_b}{(\beta+1-\alpha)_b},\\C_{bj}^{(\alpha,\beta)}&=\frac{(-1)^j}{j!}\frac{(k+2-2j)_j}{(k+2d-1-j)_j}(k-2j-\beta+1)_\beta(-\beta)_\alpha\frac{(-1)^b}{b!}\frac{(-\alpha)_b(\beta-k+2j)_b}{(\beta+1-\alpha)_b}.\end{aligned}$$

For the case $j=0$, this further reduces to

$$C_{b0}^{(\alpha,\beta)}=(k-\beta+1)_\beta(-\beta)_\alpha\frac{(-1)^b}{b!}\frac{(-\alpha)_b(\beta-k)_b}{(\beta+1-\alpha)_b},$$

and we recover the Lemma 8.2 as the particular case $j=0$ and $d=1$.

The case $\alpha\ge\beta$ can easily be obtained in a similar way to Theorem 10.14.

Theorem 10.16. *For $0\le\alpha,\beta\le k-2j$, $0\le j\le\frac{k}{2}$, let*

$$m:=\min\{j+\alpha,k-j-\beta,j+\beta,k-j-\alpha\},\qquad c:=\min\{\alpha,\beta\}.$$

Then we have

$$L^\alpha R^\beta Z_{k-2j}^{(k)}=\sum_{\substack{0\le r\le j\\0\le b\le m-r}}C_{br}^{(\alpha,\beta)}[k-j+c-\alpha-\beta-b-r,\alpha-c+b,j+c-b-r,\beta-c+b,r],\tag{10.32}$$

where and $C_{br}^{(\alpha,\beta)}:=A_{br}^{(\alpha,\beta)}B_{br}^{(\alpha,\beta)}$ is given by (10.28) and (10.29) for $\alpha\le\beta$, and by

$$A_{br}^{(\alpha,\beta)}:=A_{br}^{(\beta,\alpha)},\quad B_{br}^{(\alpha,\beta)}:=B_{br}^{(\beta,\alpha)},\qquad \alpha\ge\beta.$$

For $\beta\ge\alpha$, the constant $B_{br}^{(\alpha,\beta)}$ can be calculated from $B_{br}^{(\alpha,0)}:=1$ and the recurrence

$$B_{br}^{(\alpha,\beta)}=(k-j-\alpha-b+1-r)bB_{b-1,r}^{(\alpha,\beta-1)}-(\alpha-\beta+1+b)(j+\beta-b-r)B_{br}^{(\alpha,\beta-1)}.\tag{10.33}$$

Proof. In light of Theorem 10.14, we need only consider the case $\alpha\ge\beta$. It follows from (10.30) that $B_{br}^{(\alpha,\beta)}:=B_{br}^{(\beta,\alpha)}$, $\alpha\ge\beta$, satisfies (10.33). The formula (10.32) can be proved as in Theorem 10.14, using induction on β and by applying R to $B_{br}^{(\alpha,\beta-1)}$. $\square$

We now consider an alternative formula for the zonal polynomial of (10.18), i.e.,

$$P_{k-2j}^{(k)}:=\sum_{b+c+r=j}\frac{(-1)^r}{b!c!r!}\frac{(k+2-j-r)_r}{(k+2d-1-r)_r}[k-j-b-r,b,c,b,r],$$

where

$$[k-j-b-r,b,c,b,r] = z_1^{k-j-b-r} w_1^b \overline{z_1}^c \overline{w_1}^b \|z+jw\|^{2r} \qquad = z_1^{k-2j} |z_1|^{2c} |w_1|^{2b} \|z+jw\|^{2r}.$$

By the binomial identity, we have

$$\begin{aligned} P_{k-2j}^{(k)} &= \sum_{r=0}^{j} \frac{(-1)^r}{r!} \frac{(k+2-j-r)_r}{(k+2d-1-r)_r} z_1^{k-2j} \|z+jw\|^{2r} \frac{1}{(j-r)!} \sum_{b+c=j-r} \frac{(b+c)!}{b!c!} |z_1|^{2c} |w_1|^{2b} \\ &= z_1^{k-2j} \sum_{r=0}^{j} \frac{(-1)^r}{r!} \frac{(k+2-j-r)_r}{(k+2d-1-r)_r} \|z+jw\|^{2r} \frac{1}{(j-r)!} (|z_1|^2+|w_1|^2)^{j-r}. \end{aligned}$$

This has the structural form

$$P_{k-2j}^{(k)} = z_1^{k-2j} F(\|z+jw\|^2, |z_1|^2+|w_1|^2), \tag{10.34}$$

where F is a homogeneous polynomial of degree j with real coefficients. An elementary calculation shows that the polynomial

$$|\langle z+jw, e_1\rangle|^2 = |z_1|^2 + |w_1|^2 = z_1\overline{z_1} + w_1\overline{w_1}$$

is in the kernel of R, R^*, L and L^*, e.g.,

$$R(z_1\overline{z_1} + w_1\overline{w_1}) = \overline{w_1 z_1} - \overline{z_1 w_1} = 0.$$

Hence, by (3.7), all polynomials of the form

$$g = G(z_1\overline{z_1} + w_1\overline{w_1}, \cdots, z_d\overline{z_d} + w_d\overline{w_d}), \tag{10.35}$$

which include $\|z+jw\|^2$ and $|z_1|^2+|w_1|^2$, are in the kernel of R, R^*, L and L^*, and hence

$$T(fg) = T(f)g + fT(g) = T(f)g, \qquad T = R, R^*, L, L^*. \tag{10.36}$$

Applying this to (10.34) gives the following.

Theorem 10.17. *Let $q' = e_1$. For $d \ge 2$, $0 \le j \le \frac{k}{2}$, the zonal polynomials of Theorem 10.8 are given by*

$$L^\alpha R^\beta P_{k-2j}^{(k)} = L^\alpha R^\beta (z_1^{k-2j}) F, \qquad 0 \le \alpha, \beta \le k-2j, \tag{10.37}$$

where $F = F(\|z+jw\|^2, |z_1|^2+|w_1|^2)$ does not depend on α and β, and is given by

$$\begin{aligned} F &= \frac{(-1)^j (k-2j+2)_j}{(k+2d-1-j)_j j!} \sum_{s=0}^{j} (-j)_s \frac{(k-2j+2d-1+j)_s}{(k-2j+2)_s} \frac{1}{s!} \|z+jw\|^{2(j-s)} \bigl(|z_1|^2+|w_1|^2\bigr)^s \\ &= \frac{(-1)^j}{(k+2d-1-j)_j} \|z+jw\|^{2j} P_j^{(k-2j+1,2d-3)} \Bigl(1 - 2\frac{|z_1|^2+|w_1|^2}{\|z+jw\|^2}\Bigr), \end{aligned} \tag{10.38}$$

with $P_j^{(k-2j+1,2d-3)}$ a Jacobi polynomial.

Proof. Since the function F of (10.34) is of the form (10.35), we may apply (10.36) repeatedly, to obtain

$$L^\alpha R^\beta P_{k-2j}^{(k)} = L^\alpha R^\beta \bigl(z_1^{k-2j}\bigr) F(\|z+jw\|^2, |z_1|^2+|w_1|^2),$$

where F does not depend on α and β, and is given by

$$F = \frac{P_{k-2j}^{(k)}}{z_1^{k-2j}} = \sum_{r=0}^{j} \frac{(-1)^r}{r!} \frac{(k+2-j-r)_r}{(k+2d-1-r)_r} \|z+jw\|^{2r} \frac{1}{(j-r)!} (|z_1|^2+|w_1|^2)^{j-r}.$$

By making the change of variables $s = j - r$, we obtain

$$\begin{aligned} F &= \frac{(-1)^j (k-2j+2)_j}{(k+2d-1-j)_j j!} \sum_{s=0}^{j} (-j)_s \frac{(k-2j+2d-1+j)_s}{(k-2j+2)_s} \frac{1}{s!} \|z+jw\|^{2(j-s)} \big(|z_1|^2+|w_1|^2\big)^s \\ &= \frac{(-1)^j (k-2j+2)_j}{(k+2d-1-j)_j j!} \|z+jw\|^{2j} \sum_{s=0}^{j} (-j)_s \frac{(k-2j+2d-1+j)_s}{(k-2j+2)_s} \frac{1}{s!} \Big(\frac{|z_1|^2+|w_1|^2}{\|z+jw\|^2}\Big)^s, \end{aligned}$$

so that F can be expressed in terms of a Jacobi polynomial, i.e.

$$F = \frac{(-1)^j}{(k+2d-1-j)_j} z_1^{k-2j} \|z+jw\|^{2j} P_j^{(k-2j+1,2d-3)}\Big(1 - 2\frac{|z_1|^2+|w_1|^2}{\|z+jw\|^2}\Big). \quad \square$$

The formula (10.38) for the zonal polynomial (reproducing kernel) $P_{k-2j}^{(k)} = z_1^{k-2j} F$, involving the Jacobi polynomial, appears in [9] (Theorem 8). An explicit formula for the factor $L^\alpha R^\beta (z_1^{k-2j})$ is given by the formula (8.3) for the univariate case (replace z by z_1, etc).

By writing the zonal polynomials in the form (10.37), the squares in the table/schematic for the zonal polynomials (see Fig. 1) become essentially those for the univariate cases $\mathrm{Harm}_{k-2j}(\mathbb{H}, \mathbb{C})$, as depicted in (8.2).

11. Symmetries

The polynomials $L^\alpha R^\beta P_{k-2j}^{(k)}$ and the spaces $H_k^{(\alpha,\beta)}(\mathbb{H}^d)$ have certain natural symmetries that correspond to the symmetries of the square (see the array in Example 10.7).

Let the permutation group $\mathrm{Sym}(4)$ act on functions of four variables in the natural way, i.e.,

$$\sigma \cdot f(x_1, x_2, x_3, x_4) = f(x_{\sigma 1}, x_{\sigma 2}, x_{\sigma 3}, x_{\sigma 4}),$$

and hence on functions $f(z, w, \overline{z}, \overline{w}) \in \mathrm{Hom}_k(\mathbb{H}^d, \mathbb{C})$. There is a subgroup G of $\mathrm{Sym}(4)$ which maps $\mathrm{Harm}_k(\mathbb{H}^d, \mathbb{C})$ to itself, which is generated by the permutations

$$\sigma := (24), \quad \tau := (14)(23). \tag{11.1}$$

This group is the dihedral group of symmetries of the square

$$D_4 = \langle a, b | a^4 = b^2 = (ba)^2 = 1 \rangle, \qquad a = \sigma\tau, \quad b = \sigma,$$

and hence has order eight. By considering the action on monomials, one obtains

$$\sigma \cdot H_k^{(\alpha,\beta)}(\mathbb{H}^d, \mathbb{C}) = H_k^{(\beta,\alpha)}(\mathbb{H}^d, \mathbb{C}), \qquad \tau \cdot H_k^{(\alpha,\beta)}(\mathbb{H}^d, \mathbb{C}) = H_k^{(\alpha,k-\beta)}(\mathbb{H}^d, \mathbb{C}), \tag{11.2}$$

and so G permutes the subspaces $H_k^{(a,b)}(\mathbb{H}^d, \mathbb{C})$, where

$$(a,b) \in \{(\alpha,\beta), (\alpha,k-\beta), (k-\alpha,\beta), (k-\alpha,k-\beta), (\beta,\alpha), (\beta,k-\alpha-k), (k-\beta,\alpha), (k-\beta,k-\alpha)\},$$

via the action on the indices given by

$$\sigma\cdot(\alpha,\beta):=(\beta,\alpha), \qquad \tau\cdot(\alpha,\beta):=(\alpha,k-\beta).$$

It clear from Lemma 7.4 that these subspaces do indeed have the same dimension. The number of subspaces above can be $1,2,4,8$, depending on the position of the index (α,β) in the square array $\{0,1,\ldots,k\}^2$. The corresponding symmetries of the zonal polynomials

$$L^\alpha R^\beta P_{k-2j}^{(k)}\in H_k^{(\alpha+j,\beta+j)}(\mathbb{H}^d,\mathbb{C}), \qquad 0\le\alpha,\beta,\le k-2j, \tag{11.3}$$

are as follows.

Lemma 11.1. *(Eight symmetries) The zonal polynomials $(L^\alpha R^\beta P_{k-2j}^{(k)})_{0\le\alpha,\beta,\le k-2j}$ have the following basic symmetries corresponding to the σ and τ of (11.1)*

$$\begin{aligned} L^\alpha R^\beta P_{k-2j}^{(k)}(z,w,\overline{z},\overline{w}) &= L^\beta R^\alpha P_{k-2j}^{(k)}(z,\overline{w},\overline{z},w) \qquad (\sigma)\\ &= c_{\alpha,\beta}L^\alpha R^{k-2j-\beta}P_{k-2j}^{(k)}(\overline{w},\overline{z},w,z) \qquad (\tau),\end{aligned} \tag{11.4}$$

where $c_{\alpha,\beta}$ is a constant. The identities for the remaining nontrivial elements of G are

$$\begin{aligned} &L^\alpha R^\beta P_{k-2j}^{(k)}(z,w,\overline{z},\overline{w})\\ &\quad= c_{\alpha,\beta}L^{k-2j-\beta}R^\alpha P_{k-2j}^{(k)}(\overline{w},z,w,\overline{z}) \qquad (\sigma\tau)\\ &\quad= c_{\beta,\alpha}L^\beta R^{k-2j-\alpha}P_{k-2j}^{(k)}(w,\overline{z},\overline{w},z) \qquad (\tau\sigma)\\ &\quad= c_{\beta,\alpha}L^{k-2j-\alpha}R^\beta P_{k-2j}^{(k)}(w,z,\overline{w},\overline{z}) \qquad (\sigma\tau\sigma)\\ &\quad= c_{\alpha,\beta}c_{k-2j-\beta,\alpha}L^{k-2j-\beta}R^{k-2j-\alpha}P_{k-2j}^{(k)}(\overline{z},w,z,\overline{w}) \qquad (\tau\sigma\tau)\\ &\quad= c_{\alpha,\beta}c_{k-2j-\beta,\alpha}L^{k-2j-\alpha}R^{k-2j-\beta}Z_{k-2j}^{(k)}(\overline{z},\overline{w},z,w) \qquad (\sigma\tau\sigma\tau).\end{aligned} \tag{11.5}$$

Proof. The permutations σ and τ map zonal polynomials to zonal polynomials, and so, in light of (11.2) and (11.3), we obtain (11.4). This could also be established, with values of the constants, from the formulas for $L^\alpha R^\beta P_{k-2j}^{(k)}$ given in Theorem 10.14, or by using identities such as

$$\sigma\cdot(Rf)=Lf, \quad \sigma\cdot(Lf)=Rf, \qquad \tau\cdot(Rf)=R^*f, \quad \tau\cdot(Lf)=-Lf,$$

together with Lemma 5.5. □

The formulas in Lemma 11.1 for $P_{k-2j,a,b}^{(k)}=L^{a-j}R^{b-j}P_{k-2j}^{(k)}$, $\alpha=a-j$, $\beta=b-j$, do not have a simple formula for the constants, as the normalisation of $L^\alpha R^\beta P_{k-2j}^{(k)}$ is biased towards the (starting) polynomial $P_{k-2j}^{(k)}\in H_k^{(j,j)}(\mathbb{H}^d,\mathbb{C})$, which corresponds to a corner of the array of indices. For k even, one could start with the "centre" polynomial

$$C_{k-2j}^{(k)}=P_{k-2j,\frac{k}{2},\frac{k}{2}}^{(k)}=L^{\frac{k}{2}-j}R^{\frac{k}{2}-j}P_{k-2j}^{(k)}\in H_k^{(\frac{k}{2},\frac{k}{2})}(\mathbb{H}^d,\mathbb{C}),$$

to obtain zonal polynomials

$$L^{\max\{\alpha,\frac{k}{2}-j\}}(L^*)^{\max\{\frac{k}{2}-j-\alpha,0\}}R^{\max\{\beta,\frac{k}{2}-j\}}(R^*)^{\max\{\frac{k}{2}-j-\beta,0\}}C_{k-2j}^{(k)}.$$

By using Lemma 5.5, these can be written as

$$P^{(\alpha,\beta)}_{k,k-2j} := \frac{M_\alpha!}{\alpha!}\frac{M_\beta!}{\beta!} R^\alpha L^\beta P^{(k)}_{k-2j},$$

where

$$M_\alpha := \max\{\alpha, k-2j-\alpha\}, \qquad M_\beta := \max\{\beta, k-2j-\alpha\}.$$

The σ and τ symmetries of Lemma 11.1 then become

$$\begin{aligned} P^{(\alpha,\beta)}_{k,k-2j}(z,w,\overline{z},\overline{w}) &= P^{(\beta,\alpha)}_{k,k-2j}(z,\overline{w},\overline{z},w) \qquad (\sigma) \\ &= (-1)^\alpha P^{(\alpha,k-2j-\beta)}_{k,k-2j}(\overline{w},\overline{z},w,z) \qquad (\tau). \end{aligned} \tag{11.6}$$

12. The fine scale decomposition for left and right multiplication by $\mathbb{H}^*$

By taking the intersection of the decomposition into irreducibles for right multiplication by $\mathbb{H}^*$ (Theorem 9.1) with the corresponding one for left multiplication, we obtain the following decomposition into low dimensional subspaces. All of our decompositions, and others, can be built up from this.

Theorem 12.1. *(Fine scale decomposition) Let*

$$V^{(j_1,j_2)}_k(\mathbb{H}^d) := \ker L^* \cap \ker R^* \cap H^{(j_1,j_2)}_k(\mathbb{H}^d), \qquad 0 \le j_1, j_2 \le \frac{k}{2}.$$

Then for $d \ge 2$, we have the orthogonal direct sum

$$\operatorname{Hom}_k(\mathbb{H}^d,\mathbb{C}) = \bigoplus_{0\le j_1,j_2\le\frac{k}{2}} \bigoplus_{\substack{j_1\le a\le k-j_1\\ j_2\le b\le k-j_2}} \bigoplus_{i=0}^{\min\{j_1,j_2\}} \|\cdot\|^{2i} L^{a-j_1} R^{b-j_2} V^{(j_1-i,j_2-i)}_{k-2i}(\mathbb{H}^d), \tag{12.1}$$

and in particular

$$\operatorname{Harm}_k(\mathbb{H}^d,\mathbb{C}) = \bigoplus_{0\le j_1,j_2\le\frac{k}{2}} \bigoplus_{\substack{j_1\le a\le k-j_1\\ j_2\le b\le k-j_2}} L^{a-j_1} R^{b-j_2} V^{(j_1,j_2)}_k(\mathbb{H}^d), \tag{12.2}$$

where

$$\dim(L^{a-j_1} R^{b-j_2} V^{(j_1,j_2)}_k(\mathbb{H}^d)) = \dim(V^{(j_1,j_2)}_k(\mathbb{H}^d)).$$

Proof. We note that for $j \le \frac{k}{2}$, $j \le k-j$, so that $\min\{j,k-j\} = j$, and

$$\operatorname{Hom}_H(k-j,j) = \bigoplus_{i=0}^{j} \|\cdot\|^{2i} H(k-j-i,j-i). \tag{12.3}$$

We observe that Lemma 10.4 implies multiplication of polynomials by $\|\cdot\|^2$ commutes with the action of R, L, R^*, L^*. Thus from (12.3), we obtain

$$\begin{aligned}\mathrm{Hom}_H(k-j,j)_{k-2j} &:= \ker R^* \cap \mathrm{Hom}_H(k-j,j) \\ &= \bigoplus_{i=0}^{j} \|\cdot\|^{2i}(\ker R^* \cap H(k-j-i,j-i)),\end{aligned}$$

so that Lemma 6.5 gives the orthogonal direct sum decomposition

$$\mathrm{Hom}_k(\mathbb{H}^d,\mathbb{C}) = \bigoplus_{0\le j\le \frac{k}{2}} \bigoplus_{j\le b\le k-j} \bigoplus_{i=0}^{j} \|\cdot\|^{2i} R^{b-j}(\ker R^* \cap H(k-j-i,j-i)).$$

Similarly, we obtain the orthogonal direct sum decomposition

$$\mathrm{Hom}_k(\mathbb{H}^d,\mathbb{C}) = \bigoplus_{0\le j\le \frac{k}{2}} \bigoplus_{j\le a\le k-j} \bigoplus_{i=0}^{j} \|\cdot\|^{2i} L^{a-j}(\ker L^* \cap K(k-j-i,j-i)).$$

Thus $\mathrm{Hom}_k(\mathbb{H}^d,\mathbb{C})$ is an orthogonal direct sum of subspaces

$$\|\cdot\|^{2i_1} L^{a-j_1}(\ker L^* \cap K(k-j_1-i_1,j_1-i_1)) \cap \|\cdot\|^{2i_2} R^{b-j_2}(\ker R^* \cap H(k-j_2-i_2,j_2-i_2)).$$

In view of the uniqueness of the Fischer decomposition, these can be nonzero only if $i_1 = i_2 = i \le \min\{j_1, j_2\}$. Since L, R and $\|\cdot\|^2$ commute, the intersection above can be written

$$\|\cdot\|^{i} L^{a-j_1} R^{b-j_2}(\ker L^* \cap \ker R^* \cap K(k-j_1-i_1,j_1-i_1)) \cap H(k-j_2-i_2,j_2-i_2)),$$

which gives (12.1), with the $i=0$ terms giving (12.2). The dimension formula follows from Lemma 6.4 □

Theorem 12.1 also holds for $d=1$, in a degenerate way, with

$$V_k^{(j_1,j_2)}(\mathbb{H}) = 0, \qquad (j_1,j_2) \ne (0,0).$$

Corollary 12.2. *The decomposition of zonal polynomials for $Z = U(\mathbb{H}^d)_{q'}$, $q' = z' \in \mathbb{C}^d$, corresponding to (12.2) is*

$$\mathrm{Harm}_k(\mathbb{H}^d,\mathbb{C})^Z = \bigoplus_{0\le j\le \frac{k}{2}} \bigoplus_{j\le a,b\le k-j} \left(L^{a-j} R^{b-j} V_k^{(j,j)}(\mathbb{H}^d)\right)^Z, \tag{12.4}$$

where

$$\dim(L^{a-j} R^{b-j} V_k^{(j,j)}(\mathbb{H}^d)^Z) = 1.$$

Moreover, we have

$$H_k^{(a,b)}(\mathbb{H}^d) = \bigoplus_{\substack{0\le j_1\le m_a^{(k)} \\ 0\le j_2\le m_b^{(k)}}} L^{a-j_1} R^{b-j_2} V_k^{(j_1,j_2)}(\mathbb{H}^d). \tag{12.5}$$

Proof. The decomposition (12.4) is given in Theorem 10.8, and the decomposition (12.5) follows from (12.2) by grouping the terms $L^{a-j_1} R^{b-j_2} V_k^{(j_1,j_2)}(\mathbb{H}^d) \in H_k^{(a,b)}(\mathbb{H}^d)$. □

The dimension of $V_k^{(a,b)}(\mathbb{H}^d)$ is as follows (see [21]).

Lemma 12.3. *For $0 \le a, b \le \frac{k}{2}$, we have that*

$$\begin{aligned}\dim(V_k^{(a,b)}(\mathbb{H}^d)) &= F(k,m,M,d) + F(k,m-1,M-1,d) \\ &\quad - F(k,m-1,M,d) - F(k,m,M-1,d), \end{aligned} \tag{12.6}$$

where F is given by (7.10), and $m = \min\{a,b\}$, $M = \max\{a,b\}$. In particular,

$$\dim(V_k^{(a,b)}(\mathbb{H}^2)) = (m+1)(k-2M+1).$$

The zonal polynomials in $V_k^{(a,b)}(\mathbb{H}^d)^{U_q}$ is given by

$$\dim(V_k^{(a,b)}(\mathbb{H}^d)^{U_q}) = \begin{cases} 1, & a = b; \\ 0, & a \ne b. \end{cases}$$

For $d \ge 2$, it follows from Lemma 12.3 that all the summands in (12.1) and (12.2) are nontrivial.

Example 12.4. We have

$$V_k^{(0,0)}(\mathbb{H}^d) = H_k^{(0,0)}(\mathbb{H}^d) = \operatorname{span}\{z^\alpha : |\alpha| = k\}, \quad \dim(V_k^{(0,0)}(\mathbb{H}^d)) = \binom{k+d-1}{d-1}.$$

For $k = 2$, $d = 2$, $V_2^{(0,0)}(\mathbb{H}^2) = \operatorname{span}\{z_1^2, z_1z_2, z_2^2\}$, and

$$V_2^{(0,1)}(\mathbb{H}^2) = \operatorname{span}\{z_1\overline{w_2} - z_2\overline{w_1}\}, \quad V_2^{(1,0)}(\mathbb{H}^2) = \operatorname{span}\{z_1w_2 - z_2w_1\},$$

$$V_2^{(1,1)}(\mathbb{H}^2) = \operatorname{span}\{z_1\overline{z_1} + w_1\overline{w_1} - z_2\overline{z_2} - w_2\overline{w_2}, z_1\overline{z_2} + \overline{z_1}z_2 + w_1\overline{w_2} + \overline{w_1}w_2\}.$$

Here we can see explicitly, that the zonal polynomials for $Z = U(\mathbb{H}^2)_{e_1}$ are given by

$$V_2^{(0,0)}(\mathbb{H}^2)^Z = \operatorname{span}\{z_1^2\}, \quad V_2^{(0,1)}(\mathbb{H}^2)^Z = 0, \quad V_2^{(1,0)}(\mathbb{H}^2)^Z = 0,$$

$$V_2^{(1,1)}(\mathbb{H}^2)^Z = \operatorname{span}\{2(z_1\overline{z_1} + w_1\overline{w_1}) - \|(z,w)\|^2\}.$$

The decomposition of (12.4) involves subspaces of dimensions 3 (nine), 2 (one) and 1 (six), with

$$\dim(\operatorname{Harm}_2(\mathbb{H}^2)) = 9 \cdot 3 + 1 \cdot 2 + 6 \cdot 1 = 35.$$

13. Conclusion

The fine scale decomposition of Theorem 12.1 refines all the decompositions of the harmonic polynomials $\operatorname{Harm}_k(\mathbb{H}^d, \mathbb{C})$ under the action of a group $G \subset U(\mathbb{R}^{4d})$ that we have given or described. These can be summarised as follows:

$U(\mathbb{R}^{4d})$	(a single irreducible) [1]
$\vert$	
$U(\mathbb{C}^{2d})$	(the spaces $H(k-b,b)$) [18], [11], [22], [8]
$\vert$	
$U(\mathbb{H}^d)\times U(\mathbb{H})$	(Theorem 9.3) [23], [3]
$\vert$	
$U(\mathbb{H}^d)$	(Theorem 9.2) [6], [4]
$\vert$	
$U(\mathbb{H})$	(Theorem 9.1) [6]
$\vert$	
1	(a single homogeneous component).

Here the action of $U(\mathbb{H}^d)\times U(\mathbb{H}) = \mathrm{Sp}(d)\times\mathrm{Sp}(1)$, and similar products, is not faithful, since the real unitary scalar matrix $-I$ belongs to $U(\mathbb{H}^d)$ and $U(\mathbb{H})$, where it has the same action. One can naturally obtain irreducible decompositions by applying the given group to components of the fine scale decomposition. For example, we have the following.

Corollary 13.1. *Let $d\geq 2$. For the action given by left and right multiplication by $\mathbb{H}^*=\mathrm{Sp}(1)$, i.e., the group $G=\mathrm{Sp}(1)\times\mathrm{Sp}(1)$, we have the following orthogonal direct sum of homogeneous components*

$$\mathrm{Harm}_k(\mathbb{H}^d,\mathbb{C}) = \bigoplus_{0\leq j_1,j_2\leq\frac{k}{2}}\Big\{\bigoplus_{\substack{j_1\leq a\leq k-j_1\\ j_2\leq b\leq k-j_2}} L^{a-j_1}R^{b-j_2}V_k^{(j_1,j_2)}(\mathbb{H}^d)\Big\} = \bigoplus_{0\leq j_1,j_2\leq\frac{k}{2}} I(W_{k-2j_1,k-2j_2})^{(k)}$$
$$\cong \bigoplus_{0\leq j_1,j_2\leq\frac{k}{2}} \dim(V_k^{(j_1,j_2)}(\mathbb{H}^d))\cdot W_{k-2j_1,k-2j_2}, \tag{13.1}$$

for the irreducibles

$$W_{k-2j_1,k-2j_2}\cong \mathrm{span}_\mathbb{C}\{L^{a-j_1}R^{b-j_2}f\}_{\substack{j_1\leq a\leq k-j_1\\ j_2\leq b\leq k-j_2}},\quad f\neq 0,\quad f\in V_k^{(j_1,j_2)}(\mathbb{H}^d). \tag{13.2}$$

Proof. The orthogonal direct sums in (13.1) are immediate, and the $I(W_{k-2j_1,k-2j_2})^{(k)}$ defined is a sum of the subspaces in (13.2). It is easily seen from Theorem 9.1, and its analogue for L, that these subspaces, i.e.,

$$\mathrm{span}_\mathbb{C}\{L^\alpha R^\beta f\}_{\substack{0\leq\alpha\leq k-2j_1\\ 0\leq\beta\leq k-2j_2}},\quad f\neq 0,\quad f\in V_k^{(j_1,j_2)}(\mathbb{H}^d),$$

are invariant under left and right multiplication by $\mathbb{H}^*$, that the action is irreducible, and they are isomorphic $\mathbb{C}G$-modules. □

There has be recent work on the $\mathbb{H}$-valued slice regular functions on $\mathbb{H}^d$, e.g., [13], but it is not clear whether our results could be adapted here, as their definition is involved and the theory of $\mathbb{H}$-modules is far less developed that of $\mathbb{C}$-modules.

Finally, we observe that the central idea underlying our development is the Lie correspondence of Lemma 3.8, which allows us to replace invariance under the continuous group $\mathbb{H}^*$ (or $\mathrm{Sp}(1)$) by invariance under the finite set of operators $\{R,R^*\}$. The same development can be applied directly to (a left action) of the simply connected matrix Lie group $\mathrm{Sp}(n)=U(\mathbb{H}^d)$, see, e.g., [4].

References

[1] Sheldon Axler, Paul Bourdon, Wade Ramey, Harmonic Function Theory, second edition, Graduate Texts in Mathematics, vol. 137, Springer-Verlag, New York, 2001.

[2] Marcus Appleby, Tuan-Yow Chien, Steven Flammia, Shayne Waldron, Constructing exact symmetric informationally complete measurements from numerical solutions, J. Phys. A 51 (16) (2018) 165302.

[3] Julian Ahrens, Michael G. Cowling, Alessio Martini, Detlef Müller, Quaternionic spherical harmonics and a sharp multiplier theorem on quaternionic spheres, Math. Z. 294 (3–4) (2020) 1659–1686.

[4] Fred Brackx, Hennie De Schepper, David Eelbode, Roman Lavicka, Vladimir Soucek, Fundaments of quaternionic Clifford analysis III: Fischer decomposition in symplectic harmonic analysis, arXiv e-prints, arXiv:1404.3625, April 2014.

[5] Fred Brackx, Hennie De Schepper, David Eelbode, Roman Lavicka, Vladimir Soucek, Fischer decomposition for the symplectic group, J. Math. Anal. Appl. 458 (1) (2018) 831–848.

[6] Christine Bachoc, Gabriele Nebe, Zonal functions for the unitary groups and applications to Hermitian lattices, J. Number Theory 96 (1) (2002) 55–75.

[7] J.L. Brenner, Matrices of quaternions, Pac. J. Math. 1 (1951) 329–335.

[8] Agata Bezubik, Aleksander Strasburger, On spherical expansions of smooth $\mathbf{SU}(n)$-zonal functions on the unit sphere in $\mathbb{C}^n$, J. Math. Anal. Appl. 404 (2) (2013) 570–578.

[9] H. De Bie, F. Sommen, M. Wutzig, Plane wave formulas for spherical, complex and symplectic harmonics, J. Approx. Theory 222 (2017) 110–131.

[10] Feng Dai, Yuan Xu, Approximation Theory and Harmonic Analysis on Spheres and Balls, Springer Monographs in Mathematics, Springer, New York, 2013.

[11] G.B. Folland, Spherical harmonic expansion of the Poisson-Szegö kernel for the ball, Proc. Am. Math. Soc. 47 (1975) 401–408.

[12] Gerald B. Folland, A Course in Abstract Harmonic Analysis, Studies in Advanced Mathematics, CRC Press, Boca Raton, FL, 1995.

[13] Riccardo Ghiloni, Alessandro Perotti, Slice regular functions in several variables, Math. Z. 302 (1) (2022) 295–351.

[14] Brian Hall, Lie Groups, Lie Algebras, and Representations: An Elementary Introduction, second edition, Graduate Texts in Mathematics, vol. 222, Springer, Cham, 2015.

[15] S.G. Hoggar, Parameters of t-designs in $\mathbf{F}P^{d-1}$, Eur. J. Comb. 5 (1) (1984) 29–36.

[16] Tomoyoshi Ibukiyama, Yasutaka Ihara, On automorphic forms on the unitary symplectic group $\mathrm{Sp}(n)$ and $\mathrm{SL}_2(\mathbf{R})$, Math. Ann. 278 (1–4) (1987) 307–327.

[17] Mineo Ikeda, Noriaki Seto, On expansion theorems in terms of spherical functions for unitary group. I, Math. Jpn. 13 (1968) 149–157.

[18] Tom Koornwinder, The addition formula for Jacobi polynomials and spherical harmonics, SIAM J. Appl. Math. 25 (1973) 236–246.

[19] Casey Machen, The exponential of a quaternionic matrix, Rose-Hulman Undergrad. Math J. 12 (2) (2011) 29–36.

[20] Mozhgan Mohammadpour, The construction of complex and quaternionic spherical designs, PhD thesis, University of Auckland, 2022, Available at https://researchspace.auckland.ac.nz/handle/2292/64987.

[21] Mozhgan Mohammadpour, Shayne Waldron, An Explicit Construction of the Unitarily Invariant Quaternionic Polynomial Spaces on the Sphere, 2024.

[22] Walter Rudin, Function Theory in the Unit Ball of $\mathbf{C}^n$, Grundlehren der Mathematischen Wissenschaften (Fundamental Principles of Mathematical Science), vol. 241, Springer-Verlag, New York-Berlin, 1980.

[23] R.T. Smith, The spherical representations of groups transitive on S^n, Indiana Univ. Math. J. 24 (1974/1975) 307–325.

[24] A. Sudbery, Quaternionic analysis, Math. Proc. Camb. Philos. Soc. 85 (2) (1979) 199–224.

[25] Gert Vegter, The apolar bilinear form in geometric modeling, Math. Comput. 69 (230) (2000) 691–720.

[26] Shayne F.D. Waldron, An Introduction to Finite Tight Frames. Applied and Numerical Harmonic Analysis, Birkhäuser/Springer, New York, 2018.

[27] Shayne Waldron, A variational characterisation of projective spherical designs over the quaternions, arXiv e-prints, arXiv:2011.08439, November 2020.

[28] Shayne Waldron, Tight frames over the quaternions and equiangular lines, preprint, 2020.

[29] Doron Zeilberger, Chu's 1303 identity implies Bombieri's 1990 norm-inequality (via an identity of Beauzamy and Dégot), Am. Math. Mon. 101 (9) (1994) 894–896.