Group frames

Shayne Waldron

Abstract The prototypical example of a tight frame: the *Mercedes–Benz frame* can be obtained as the orbit of a single vector under the action of the group generated by rotation by $\frac{2\pi}{3}$, or the dihedral group of symmetries of the triangle. Many frames used in applications are constructed in this way, often as the orbit of a single vector (akin to a mother wavelet). Most notable are the *harmonic frames* (finite abelian groups) used in signal analysis, and the equiangular *Heisenberg frames*, or *SIC-POVMs*, (discrete Heisenberg group) used in quantum information theory. Other examples include tight frames of multivariate orthogonal polynomials sharing symmetries of the weight function, and the *highly symmetric tight frames* which can be viewed as the vertices of highly regular polytopes. We will describe the basic theory of such *group frames*, and some of the constructions that have been found so far.

Key words: Group frame, *G*–frame, harmonic frames, SIC-POVM, Heisenberg frame, highly symmetric tight frame, symmetry group of a frame, Heisenberg frame, group matrix, unitary representation, equiangular frames, Zauner's conjecture.

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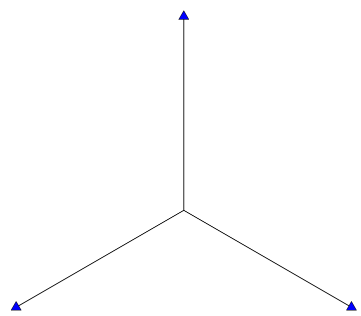
1 The symmetries of a frame (its dual and complement)

The *symmetries* of the Mercedes–Benz frame



are those rotations and reflections (unitary maps) which permute its vectors. We now formalise this idea, with the key features of the *symmetry group* (see [19] for full proofs) being:

- It is defined for *all* finite frames as a group of permutations on the index set.
- It is simple to calculate from the Gramian of the canonical tight frame.
- The symmetry groups of similar frames are equal. In particular, a frame, its dual frame and canonical tight frame have the same symmetry group.
- The symmetry group of various combinations of frames, such as tensor products and direct sums, are related to those of the constituent frames in a natural way.
- The symmetry group of a frame and its complementary frame are equal.

Let S_M be the (symmetric group of) permutations on $\{1,2,\ldots,M\}$, and $\mathrm{GL}(\mathscr{H})$ be the (general linear group of) linear maps $\mathscr{H} \to \mathscr{H}$.

Definition 1. The **symmetry group** of a finite frame $\Phi = (\varphi_j)_{j=1}^M$ for $\mathscr{H} = \mathbb{F}^N$ is

$$\mathrm{Sym}(\Phi) := \{\sigma \in S_M : \exists L_\sigma \in \mathrm{GL}(\mathscr{H}) \text{ with } L_\sigma \varphi_j = \varphi_{\sigma j},\ j = 1,\ldots,M\}.$$

Let Φ^{can} denote the canonical tight frame $(\Phi\Phi^*)^{-1/2}\Phi$ of Φ.

Theorem 1. *If Φ and Ψ are similar frames, i.e., $\Phi = Q\Psi$, $Q \in \mathrm{GL}(\mathscr{H})$, or are complementary frames, i.e., $G_{\Phi^{\mathrm{can}}} + G_{\Psi^{\mathrm{can}}} = Id$, then*

$$\mathrm{Sym}(\Psi) = \mathrm{Sym}(\Phi).$$

In particular, a frame, its dual frame and its canonical tight frame have the same symmetry group.

Proof. It suffices to show one inclusion. Suppose $\sigma \in \mathrm{Sym}(\Phi)$, i.e., $L_\sigma \varphi_j = \varphi_{\sigma j}$, $\forall j$. Since $\varphi_j = Q\psi_j$, this gives $Q^{-1}L_\sigma Q\psi_j = \psi_{\sigma j}$, $\forall j$, i.e., $\sigma \in \mathrm{Sym}(\Psi)$. $\square$

Example 1. Let Φ be the Mercedes–Benz frame. Since its vectors add to zero, $\Psi = ([1],[1],[1])$ is the complementary frame for $\mathbb{R}$. Clearly, $\mathrm{Sym}(\Psi) = S_3$, and so $\mathrm{Sym}(\Phi) = S_3$ (which is isomorphic to the dihedral group of triangular symmetries).

Since a finite frame Φ is determined up to similarity by $G_{\Phi^{\mathrm{can}}}$, the Gramian of the canonical tight frame, it is possible to compute $\mathrm{Sym}(\Phi)$ from $G_{\Phi^{\mathrm{can}}}$. This is most easily done as follows:

Proposition 1. *Let Φ be a finite frame. Then*

$$\sigma \in \mathrm{Sym}(\Phi) \quad \Longleftrightarrow \quad P_\sigma^* G_{\Phi^{\mathrm{can}}} P_\sigma = G_{\Phi^{\mathrm{can}}},$$

where P_σ is the permutation matrix given by $P_\sigma e_j = e_{\sigma j}$.

Since $\mathrm{Sym}(\Phi)$ is a subgroup of S_M, it follows there are **maximally symmetric** frames of M vectors in $\mathbb{F}^N$, i.e., those with the largest possible symmetry groups.

Example 2. The M equally spaced vectors in $\mathbb{R}^2$ have the dihedral group of order $2M$ as symmetries. This is not always the most symmetric frame of M vectors in $\mathbb{C}^2$, e.g., if M is even, the (harmonic) tight frame given by the M distinct vectors

$$\left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} \omega \\ -\omega \end{pmatrix}, \begin{pmatrix} \omega^2 \\ \omega^2 \end{pmatrix}, \begin{pmatrix} \omega^3 \\ -\omega^3 \end{pmatrix}, \begin{pmatrix} \omega^4 \\ \omega^4 \end{pmatrix}, \ldots \begin{pmatrix} \omega^{M-2} \\ \omega^{M-2} \end{pmatrix}, \begin{pmatrix} \omega^{M-1} \\ -\omega^{M-1} \end{pmatrix} \right\}, \qquad \omega := e^{\frac{2\pi i}{M}}$$

has a symmetry group of order $\frac{1}{2}M^2$ (see [10] for details).

Example 3. The most symmetric tight frames of 5 vectors in $\mathbb{R}^3$ are as follows

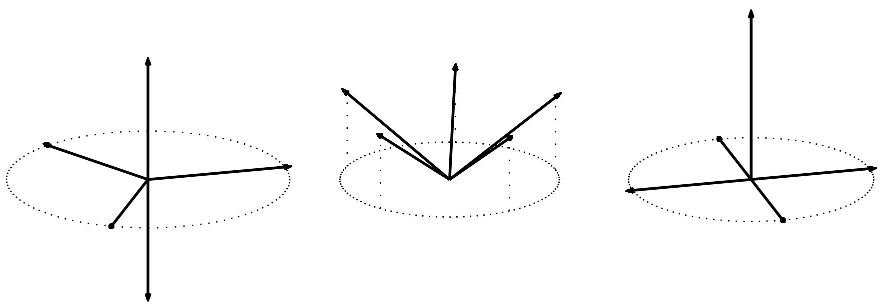


Fig. 1 The most symmetric tight frames of five distinct nonzero vectors in $\mathbb{R}^3$. The vertices of the trigonal bipyramid (12 symmetries), five equally spaced vectors lifted (10 symmetries), and four equally spaced vectors and one orthogonal (8 symmetries).

The symmetry group of a combination of frames behaves as one would expect:

Proposition 2. *The symmetry groups of a finite frame satisfy*

1. $\mathrm{Sym}(\Phi) \times \mathrm{Sym}(\Psi) \subset \mathrm{Sym}(\Phi \cup \Psi)$ *(union of frames)*

2. $\mathrm{Sym}(\Phi) \times \mathrm{Sym}(\Psi) \subset \mathrm{Sym}(\Phi \otimes \Psi)$ *(tensor product)*
3. $\mathrm{Sym}(\Phi) \cap \mathrm{Sym}(\Psi) \subset \mathrm{Sym}(\Phi \oplus \Psi)$ *(direct sum)*

Here

$$\Phi \cup \Psi := \left(\begin{pmatrix} \varphi_j \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \psi_k \end{pmatrix} \right), \qquad \Phi \otimes \Psi = (\varphi_j \otimes \psi_k),$$

$$\Phi \oplus \Psi := \left(\begin{pmatrix} \varphi_j \\ \psi_k \end{pmatrix} \right), \quad \text{where } \textstyle\sum_j \langle f, \varphi_j \rangle \psi_j = 0, \forall f.$$

Since linear maps are determined by their action on a spanning set, it follows that if $\sigma \in \mathrm{Sym}(\Phi)$, then there is a unique $L_\sigma \in \mathrm{GL}(\mathscr{H})$ with $L_\sigma f_j = f_{\sigma j}$, $\forall j$. Further,

$$\mathrm{Sym}(\Phi) \to \mathrm{GL}(\mathscr{H}) : \sigma \mapsto L_\sigma \tag{1}$$

is a group homomorphism, i.e., a *representation* of $G = \mathrm{Sym}(\Phi)$. If the symmetry group acts transitively on Φ under this action, i.e., Φ is the orbit of any one vector, e.g., the Mercedes–Benz frame, then we have what is called a *G–frame*.

2 Representations and G–frames

The Mercedes–Benz frame is the orbit under its symmetry group of a single vector. Formally, the symmetry group is a group of permutations (an abstract group) which acts as unitary transformations. This is a fundamental notion in abstract algebra:

Definition 2. A **representation** of a finite group G is a group homomorphism

$$\rho : G \mapsto \mathrm{GL}(\mathscr{H}),$$

i.e., a linear action of G on $\mathscr{H} = \mathbb{F}^N$, usually abbreviated $gv = \rho(g)v$, $v \in \mathscr{H}$.

Representations are a convenient way to study groups which appear as linear transformations, whilst being able to appeal to abstract group theory (cf. [12]).

Example 4. If Φ is a frame, then we have already observed that the action of $\mathrm{Sym}(\Phi)$ on $\mathscr{H}$ given by (1) is a representation. If Φ is *tight*, then this action is *unitary*. We will build this into our definition of a *group frame*.

Definition 3. Let G be finite group. A **group frame** or **G–frame** for $\mathscr{H}$ is a frame $\Phi = (\varphi_g)_{g \in G}$ for which there exists a unitary representation $\rho : G \to \mathscr{U}(\mathscr{H})$ with

$$g\varphi_h := \rho(g)\varphi_h = \varphi_{gh}, \qquad \forall g, h \in G.$$

This definition implies that a G–frame Φ is the orbit of a single vector $v \in \mathscr{H}$, i.e.,

$$\Phi = (gv)_{g \in G},$$

and so is an *equal–norm* frame.

Example 5. An early example of group frames is the vertices of the *regular M–gon* or the *platonic solids*. These were some of the first examples of frames considered (see [3]). The *highly symmetric tight frames* (see §7) are a variation on this theme.

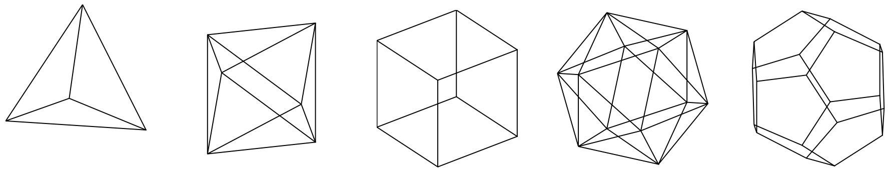


Fig. 2 The vertices of the platonic solids are examples of group frames.

In the remaining sections, we outline the basic properties and constructions for G–frames. In particular, we will see:

- There is a *finite* number of G–frames of M vectors in $\mathbb{F}^N$ for abelian groups G. These are known as *harmonic frames* (see §5)
- There is an *infinite* number of G–frames of M vectors in $\mathbb{F}^N$ for nonabelian G. Most notably, the *Heisenberg frames* (see §9) of $M = N^2$ vectors in $\mathbb{C}^N$, which provide equiangular tight frames with the maximal number of vectors.

3 Group matrices and the Gramian of a G–frame

Since the representation defining a G–frame is unitary, i.e.,

$$\rho(g)^* = \rho(g)^{-1} = \rho(g^{-1}), \qquad \text{so that } g^{-1}v = g^*v,$$

the Gramian of a G–frame $\Phi = (\varphi_g)_{g\in G} = (gv)_{g\in G}$ has a special form:

$$\langle \varphi_g, \varphi_h \rangle = \langle gv, hv \rangle = \langle v, g^*hv \rangle = \langle v, g^{-1}hv \rangle = \eta(g^{-1}h), \qquad \text{where } \eta : G \to \mathbb{F}.$$

Thus the Gramian of a G–frame is a **group matrix** or G**–matrix**, i.e., a matrix A, with entries indexed by elements of a group G, which has the form

$$A = [\eta(g^{-1}h)]_{g,h\in G}.$$

One important consequence of the fact the Gramian of a G–frame is a group matrix is that it has a small number of angles: $\{\eta(g) : g \in G\}$, which makes them good candidates for equiangular tight frames (see §9). We have the characterisation ([18]):

Theorem 2. *Let G be a finite group. Then $\Phi = (\varphi_g)_{g\in G}$ is a G–frame (for its span $\mathscr{H}$) if and only if its Gramian G_Φ is a G–matrix.*

Proof. If Φ is a G–frame, then we observed that its Gramian is a G–matrix.

Conversely, suppose that the Gramian of a frame Φ for $\mathscr{H}$ is a G–matrix. Let $\tilde{\Phi} = (\tilde{\phi}_g)_{g\in G}$ be the dual frame, so that

$$f = \sum_{g\in G} \langle f, \tilde{\phi}_g\rangle \phi_g, \qquad \forall f \in \mathscr{H}. \tag{2}$$

For each $g \in G$, define a linear operator $U_g : \mathscr{H} \to \mathscr{H}$ by

$$U_g(f) := \sum_{h_1\in G} \langle f, \tilde{\phi}_{h_1}\rangle \phi_{gh_1}, \qquad \forall f \in \mathscr{H}.$$

Since $\mathrm{Gram}(\Phi) = [\langle \phi_h, \phi_g\rangle]_{g,h\in G}$ is a G–matrix, we have

$$\langle \phi_{gh_1}, \phi_{gh_2}\rangle = \nu((gh_2)^{-1}gh_1) = \nu(h_2^{-1}h_1) = \langle \phi_{h_1}, \phi_{h_2}\rangle. \tag{3}$$

It follows from (2) and (3) that U_g is unitary by the calculation

$$\begin{aligned}
\langle U_g(f_1), U_g(f_2)\rangle &= \langle \sum_{h_1\in G} \langle f_1, \tilde{\phi}_{h_1}\rangle \phi_{gh_1}, \sum_{h_2\in G} \langle f_2, \tilde{\phi}_{h_2}\rangle \phi_{gh_2}\rangle \\
&= \sum_{h_1\in G}\sum_{h_2\in G} \langle f_1, \tilde{\phi}_{h_1}\rangle \overline{\langle f_2, \tilde{\phi}_{h_2}\rangle} \langle \phi_{gh_1}, \phi_{gh_2}\rangle \\
&= \sum_{h_1\in G}\sum_{h_2\in G} \langle f_1, \tilde{\phi}_{h_1}\rangle \overline{\langle f_2, \tilde{\phi}_{h_2}\rangle} \langle \phi_{h_1}, \phi_{h_2}\rangle \\
&= \langle \sum_{h_1\in G} \langle f_1, \tilde{\phi}_{h_1}\rangle \phi_{h_1}, \sum_{h_2\in G} \langle f_2, \tilde{\phi}_{h_2}\rangle \phi_{h_2}\rangle = \langle f_1, f_2\rangle.
\end{aligned}$$

Similarly, we have

$$U_g\phi_h = \sum_{h_1\in G} \langle \phi_h, \tilde{\phi}_{h_1}\rangle \phi_{gh_1} = \sum_{h_1\in G} \langle \phi_{gh}, \tilde{\phi}_{gh_1}\rangle \phi_{gh_1} = \phi_{gh}.$$

This implies $\rho : G \to \mathscr{U}(\mathscr{H}) : g \mapsto U_g$ is a group homomorphism, since

$$U_{g_1g_2}\phi_h = \phi_{g_1g_2h} = U_{g_1}\phi_{g_2h} = U_{g_1}U_{g_2}\phi_h, \qquad \mathscr{H} = \mathrm{span}\{\phi_h\}_{h\in G}.$$

Thus ρ is a representation of G with

$$\rho(g)\phi_h = \phi_{gh}, \qquad \forall g, h \in G,$$

i.e., Φ is a G–frame for $\mathscr{H}$. $\square$

4 The characterisation of all tight G–frames

A complete characterisation of which G–frames are tight, i.e., which orbits $(gv)_{g\in G}$ under a unitary action of G give a tight frame, was given in [17]. Before stating the general theorem, we give a special case with an instructive proof.

Theorem 3. *Let $\rho : G \to \mathscr{U}(\mathscr{H})$ be a unitary representation, which is irreducible, i.e.,*

$$\operatorname{span}\{gv : g \in G\} = \mathscr{H}, \qquad \forall v \in \mathscr{H}, v \neq 0.$$

Then every orbit $\Phi = (gv)_{g\in G}$, $v \neq 0$ is a tight frame.

Proof. Let $v \neq 0$, so that $\Phi = (gv)_{g\in G}$ is a frame. Recall the frame operator S_Φ is positive definite, so there is an eigenvalue $\lambda > 0$ with corresponding eigenvector w. Since the action is unitary, we calculate

$$S_\Phi(gw) = \sum_{h\in G} \langle gw, hv\rangle hv = g \sum_{h\in G} \langle w, g^{-1}hv\rangle g^{-1}hv = gS_\Phi(w) = \lambda(gw),$$

so that $S_\Phi = \lambda(Id)$ on $\operatorname{span}\{gw : g \in G\} = \mathscr{H}$, i.e., Φ is tight. □

Example 6. The symmetry groups of the five platonic solids acting on $\mathbb{R}^3$ as unitary transformations give irreducible representations, as do the dihedral groups acting on $\mathbb{R}^2$. Thus the vertices of the platonic solids and the M equally spaced vectors in $\mathbb{R}^2$ are tight G–frames.

For a given representation, if there exists a G–frame $\Phi = (gv)_{g\in G}$, i.e., $\operatorname{span}\{gv : g \in G\} = \mathscr{H}$, then the canonical tight frame is a tight G–frame. To describe all such tight G–frames, we need a little more terminology.

Definition 4. Let G be a finite group. We say $\mathscr{H}$ is an $\mathbb{F}G$–module if there is a unitary action $(g, v) \mapsto gv$ of G on $\mathscr{H}$, i.e., a representation $G \to \mathscr{U}(\mathscr{H})$.

A linear map $\sigma : V_j \to V_k$ between $\mathbb{F}G$–modules is said to be an $\mathbb{F}G$**–homomorphism** if $\sigma g = g\sigma$, $\forall g \in G$, and an $\mathbb{F}G$**–isomorphism** if σ is a bijection. An $\mathbb{F}G$–module is **irreducible** if the corresponding representation is, and it is **absolutely irreducible** if it is irreducible when thought of as a $\mathbb{C}G$–module in the natural way.

We can now generalise Theorem 3.

Theorem 4. *Let G be a finite group which acts on $\mathscr{H}$ as unitary transformations, and*

$$\mathscr{H} = V_1 \oplus V_2 \oplus \cdots \oplus V_m$$

be an orthogonal direct sum of irreducible $\mathbb{F}G$–modules for which repeated summands are absolutely irreducible. Then $\Phi = (gv)_{g\in G}$, $v = v_1 + \cdots + v_m$, $v_j \in V_j$ is a tight G–frame if and only if

$$\frac{\|v_j\|^2}{\|v_k\|^2} = \frac{\dim(V_j)}{\dim(V_k)}, \qquad \forall j,k,$$

and $\langle \sigma v_j, v_k \rangle = 0$ *when* V_j *is* $\mathbb{F}G$*–isomorphic to* V_k *via* $\sigma : V_j \to V_k$*. By Schur's Lemma there is at most one* σ *to check.*

This result is readily applied, indeed if there is G–frame, then there is a tight one:

Proposition 3. *Let G be a finite group which acts on $\mathscr{H}$ as unitary transformations. If there is a $v \in \mathscr{H}$ for which $(gv)_{g\in G}$ is frame, i.e., spans $\mathscr{H}$, then the associated canonical tight frame is a tight G–frame for $\mathscr{H}$.*

This can be used as an alternative way to construct tight G–frames, but requires calculation of the square root of the frame operator.

Example 7. One situation where Theorem 4 applies is to orthogonal polynomials of several variables for a weight function with some symmetries G, e.g., the inner product on bivariate polynomials given by integration over a triangle. By analogy with the univariate orthogonal polynomials, the orthogonal polynomials of degree k in N variables are those polynomials of degree k which are orthogonal to all the polynomials of degree $< k$. It is natural to seek a G–invariant tight frame for this space of dimension $\binom{k+N-1}{N-1}$. Using Theorem 4, G–invariant tight frames with one orbit, i.e., G–frames, can be constructed, e.g., [17] gives an orthonormal basis for the quadratic orthogonal polynomials on the triangle (with constant weight), which is invariant under the action of the dihedral group of symmetries of the triangle.

Example 8. For G abelian, all irreducible representations are one dimensional, and it follows that there are only finitely many tight G–frames which can be constructed from these so called "characters". We discuss the resulting *harmonic frames* next.

5 Harmonic frames

The $M \times M$ **Fourier matrix**

$$\frac{1}{\sqrt{M}} \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega & \omega^2 & \cdots & \omega^{M-1} \\ 1 & \omega^2 & \omega^4 & \cdots & \omega^{2(M-1)} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & \omega^{M-1} & \omega^{2(M-1)} & \cdots & \omega^{(M-1)(M-1)} \end{bmatrix}, \qquad \omega := e^{\frac{2\pi i}{M}} \tag{4}$$

is a unitary matrix, and so its columns (or rows) form an orthonormal basis for $\mathbb{C}^M$.

Since the projection of an orthonormal basis is a tight frame, an equal norm tight frame for $\mathbb{C}^M$ can be obtained as the columns of any submatrix obtained by taking N rows of the Fourier transform matrix. Tight frames of this type are the most commonly used in applications, due to their simplicity of construction and flexibility (various choices for the rows can be made). They date back at least to [9], early applications include [8], [11], and have been called *harmonic* or *geometrically uniform tight frames*. They provide a nice example of unit–norm tight frames:

Proposition 4. *Equal–norm tight frames of $M \geq N$ vectors in $\mathbb{C}^N$ exist. Indeed, harmonic ones can be constructed by taking any N rows of the Fourier matrix (4).*

For G an abelian group, the irreducible representations are 1–dimensional, and are usually called (**linear**) **characters** $\xi : G \to \mathbb{C}$. If $G = \mathbb{Z}_M$, the cyclic group of order M, then the M characters are

$$\xi_j : k \mapsto (\omega^j)^k, \qquad j \in \mathbb{Z}_M,$$

i.e.. the rows (or columns) of the Fourier matrix (4). Thus it follows from Theorem 4, that all $\mathbb{Z}_M$–frames for $\mathbb{C}^N$ are obtained by taking N rows (or columns) of the Fourier transform matrix. We now present the general form of this result.

Let G be a finite abelian group of order M, and $\hat{G}$ be the **character group**, i.e., the set of M characters of G which forms a group under pointwise multiplication. The groups G and $\hat{G}$ are isomorphic, which is easily seen for $G = \mathbb{Z}_M$, though not in a canonical way. The **character table** of G is the table with rows given by the characters of G. Thus the Fourier matrix is, up to a normalising factor, the character table of $\mathbb{Z}_M$, and taking N rows corresponds to taking n characters, or taking N columns corresponds to restricting the characters to N elements of $\mathbb{Z}_M$.

Definition 5. Let G be a finite abelian group of order M. We call the G–frame for $\mathbb{C}^N$ obtained by taking N rows or columns of the character table of G, i.e.,

$$\Phi = \big((\xi_j(g))_{j=1}^N\big)_{g \in G}, \quad \xi_1, \ldots, \xi_N \in \hat{G},$$

$$\text{or} \quad \Phi = \big((\xi(g_j))_{j=1}^N\big)_{\xi \in \hat{G}}, \quad g_1, \ldots, g_N \in G,$$

a **harmonic frame**.

It is easy to verify that the frames given in this definition are G and $\hat{G}$ frames, respectively. We now characterise the G–frames for G abelian (see [17] for details).

Theorem 5. *Let Φ be an equal–norm finite tight frame for $\mathbb{C}^N$. Then the following are equivalent:*

1. *Φ is a G–frame, where G is an abelian group.*
2. *Φ is harmonic (obtained from the character table of G).*

Since there is a *finite* number of abelian groups of order M, we conclude:

Corollary 1. *Fix $M \geq N$. There is a finite number of tight frames of M vectors for $\mathbb{C}^N$ (up to unitary equivalence) which are given by the orbit of an abelian group of $N \times N$ matrices, namely the harmonic frames.*

Example 9. Taking the second and last rows of (4) gives the following harmonic frame for $\mathbb{C}^2$

$$\Phi = \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} \omega \\ \overline{\omega} \end{bmatrix}, \begin{bmatrix} \omega^2 \\ \overline{\omega}^2 \end{bmatrix}, \ldots, \begin{bmatrix} \omega^{M-1} \\ \overline{\omega}^{M-1} \end{bmatrix} \right).$$

This is unitarily equivalent to the M equally spaced unit vectors in $\mathbb{R}^2$, via

$$U := \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix}, \qquad \frac{1}{\sqrt{2}}U\begin{bmatrix} \omega^j \\ \overline{\omega}^j \end{bmatrix} = \begin{bmatrix} \cos\frac{2\pi j}{n} \\ \sin\frac{2\pi j}{n} \end{bmatrix}, \quad \forall j.$$

By taking rows in complex conjugate pairs, as in the example above, and the row of 1's when N is odd, we get:

Corollary 2. *There exists a real harmonic frame of $M \geq N$ vectors for $\mathbb{R}^N$.*

Example 10. The smallest noncyclic abelian group is $\mathbb{Z}_2 \times \mathbb{Z}_2$. Its character table can be calculated as the *Kronecker product* of that for $\mathbb{Z}_2$ with itself, giving

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}.$$

Taking any pair of the last three rows gives the harmonic frame

$$\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\},$$

of four equally spaced vectors in $\mathbb{R}^2$, which is also given by $\mathbb{Z}_4$ (see Ex. 9). Taking the first row and any other gives two copies of an orthogonal basis.

Thus, harmonic frames may be given by the character tables of different abelian groups; frames which arise from cyclic groups are called **cyclic** harmonic frames. There exist harmonic frames of M vectors which are *not* cyclic. These seem to be common (see Table 1 for when noncyclic abelian groups of order M exist).

Table 1 The numbers of inequivalent *non*cyclic, *cyc*lic harmonic frames of $M \leq 35$ distinct vectors for $\mathbb{C}^N$, $N = 2, 3, 4$ when a nonabelian group of order M exists.

$N = 2$

M	non	cyc	total
4	0	3	3
8	1	7	8
9	1	6	7
12	2	13	15
16	4	13	17
18	2	18	20
20	3	19	22
24	6	27	33
25	1	15	16
27	3	18	21
28	4	25	29
32	9	25	34

$N = 3$

M	non	cyc	total
4	0	3	3
8	5	16	21
9	3	15	18
12	11	57	68
16	28	74	102
18	19	121	140
20	29	137	166
24	89	241	330
25	8	115	123
27	33	159	192
28	57	255	312
32	158	278	436

$N = 4$

M	non	cyc	total
4	0	1	1
8	8	21	29
9	5	23	28
12	30	141	171
16	139	228	367
18	80	494	574
20	154	622	776
24	604	1349	1953
25	37	636	673
27	202	973	1175
28	443	1697	2140
32	1379	2152	3531

The calculations in Table 1 come from [10]. Even more efficient algorithms for calculating the numbers of harmonic frames (up to unitary equivalence) can be based on the following result (see [5] for full details).

Definition 6. We say that subsets J and K of a finite group G are **multiplicatively equivalent** if there is an automorphism $\sigma : G \to G$ for which $K = \sigma(J)$.

Definition 7. We say that two G–frames Φ and Ψ are **unitarily equivalent via an automorphism** if

$$\varphi_g = cU\psi_{\sigma g}, \qquad \forall g \in G,$$

where $c > 0$, U is unitary, and $\sigma : G \to G$ is an automorphism.

Theorem 6. *Let G be a finite abelian group, $J,K \subset G$. The following are equivalent*

1. *The subsets J and K are multiplicatively equivalent.*
2. *The harmonic frames given by J, K are unitarily equivalent via an automorphism.*

To make effective use of this result, it is convenient to have:

Theorem 7. *([5]) Let G be an abelian group of order M, and $\Phi = \Phi_J = (\xi|_J)_{\xi\in\hat{G}}$ be the harmonic frame of M vectors for $\mathbb{C}^N$ given by $J \subset G$, where $|J| = N$. Then*

- *Φ has distinct vectors if and only if J generates G.*
- *Φ is a real frame if and only if J is closed under taking inverses.*
- *Φ is a lifted frame if and only if the identity is an element of J.*

Example 11. Seven vectors in $\mathbb{C}^3$. For $G = \mathbb{Z}_7$, the seven multiplicative equivalence classes of subsets of size three have representatives

$$\{1,2,6\},\ \{1,2,3\},\ \{0,1,2\},\ \{0,1,3\},\ \{1,2,5\} \quad \text{(class size 6)}$$

$$\{0,1,6\} \quad \text{(class size 3)} \qquad \{1,2,4\} \quad \text{(class size 2)}.$$

Each gives an harmonic frame of distinct vectors (nonzero elements generate G). None of these are unitarily equivalent since their angles are different (see Fig. 3).

Example 12. For $G = \mathbb{Z}_8$ there are 17 multiplicative equivalence classes of subsets of 3 elements. Only two of these give frames with the same angles, namely

$$\big\{\{1,2,5\},\{3,6,7\}\big\}, \qquad \big\{\{1,5,6\},\{2,3,7\}\big\}.$$

The common angle multiset is

$$\{-1,i,i,-i,-i,-2i-1,2i-1\}.$$

These frames are unitarily equivalent, but not via an automorphism.

Due to examples such as this, there is not a complete description of all harmonic frames up to unitary equivalence. There is ongoing work to classify the cyclic harmonic frames. These are the building blocks for all harmonic frames, since abelian groups are products of cyclic groups, and we have the following (see [19]):

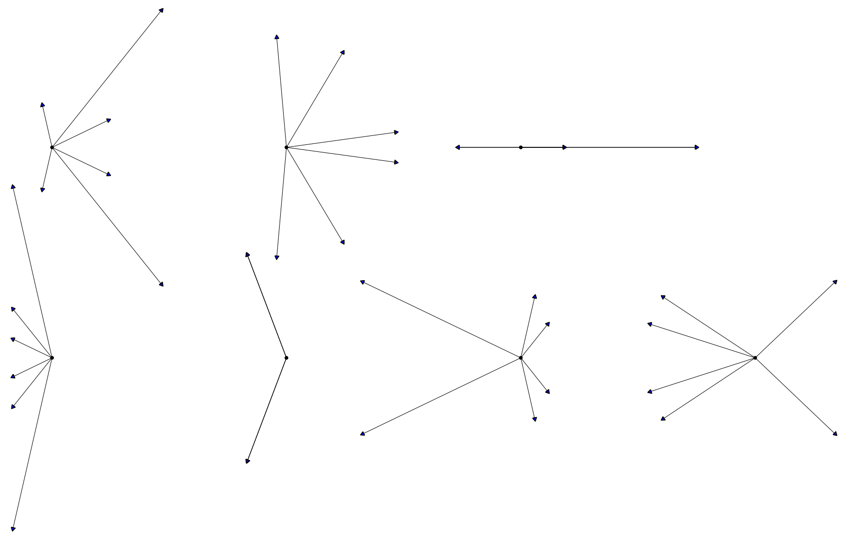


Fig. 3 The angle sets $\{\langle \varphi_0, \varphi_j \rangle : j \in G, j \neq 0\} \subset \mathbb{C}$ of the seven inequivalent harmonic frames of 7 vectors in $\mathbb{C}^3$. Note one is real, and three are equiangular.

Theorem 8. *Harmonic frames can be combined as follows:*

- *The direct sum of disjoint harmonic frames is a harmonic frame.*
- *The tensor product of harmonic frames is a harmonic frame.*
- *The complement of a harmonic frame is a harmonic frame.*

6 Equiangular harmonic frames and difference sets

We have seen in Example 11 that there exist harmonic frames which are equiangular. These are characterised by the existence of a *difference set* for an abelian group, which leads to some infinite families of equiangular tight frames.

Definition 8. An N element subset J of a finite group G of order M is said to be an (M,N,λ)**–difference set** if every nonidentity element of G can be written as a difference $a-b$ of two elements $a,b \in J$ in exactly λ ways.

Equiangular harmonic frames are in 1–1 correspondence with difference sets:

Theorem 9. *([20]) Let G be an abelian group of order M. Then the frame of M vectors for* $\mathbb{C}^N$ *obtained by restricting the characters of G to* $J \subset G$, $|J| = N$ *is an equiangular tight frame if and only if J is an* (M,N,λ)*–difference set for G.*

The parameters of a difference set satisfy

$$1 \leq \lambda = \frac{N^2 - N}{M - 1},$$

and so an equiangular harmonic frame of M vectors for $\mathbb{C}^N$ satisfies

$$M \leq N^2 - N + 1.$$

The cyclic case has been used in applications, see, e.g., [21], [13].

Example 13. For $G = \mathbb{Z}_7$ three of the seven harmonic frames in Example 11 are equiangular, i.e., the ones given by the (multiplicatively inequivalent) difference sets

$$\{1,2,4\}, \qquad \{1,2,6\}, \qquad \{0,1,3\}.$$

Example 14. The *La Jolla Difference Set Repository*

`http://www.ccrwest.org/diffsets/diff_sets/`

has numerous examples of difference sets.

7 Highly symmetric tight frames (and finite reflection groups)

For G abelian, we have seen there are *finitely* many G–frames. For G *nonabelian*, there are infinitely many. This follows from Theorem 4, but is most easily understood by an example. Let $G = D_3$ be the dihedral group of symmetries of the triangle ($|G| = 6$), acting on $\mathbb{R}^2$, so as to express the Mercedes–Benz frame as the orbit of a vector v which is fixed by a reflection. If v is not fixed by a reflection, then its orbit is a tight frame (by Theorem 3), and it is easily seen that infinitely many unitarily inequivalent tight D_3–frames of six distinct vectors for $\mathbb{R}^2$ can be obtained in this way (see Fig. 4).

All is not lost! We now consider two ways in which a finite class of G–frames can be obtained from a nonabelian (abstract) group G. The first seeks to identify the distinguishing feature of the Mercedes–Benz frame amongst the possibilities of indicated by Fig. 4, and the second (§8) generalises the notion of a harmonic frame.

Motivated by the Mercedes–Benz example:

Definition 9. A finite frame Φ of distinct vectors is **highly symmetric** if the action of its symmetry group $\mathrm{Sym}(\Phi)$ is irreducible, transitive, and the stabiliser of any one vector (and hence all) is a nontrivial subgroup which fixes a space of dimension exactly one.

Example 15. The standard orthonormal basis $\{e_1,\ldots,e_N\}$ is not a highly symmetric tight frame for $\mathbb{F}^N$, since its symmetry group fixes the vector $e_1+\cdots+e_N$. However, the vertices of the regular simplex always are (the Mercedes–Benz frame is the case $N = 2$). Since both of these frames are harmonic, we conclude that a harmonic frame may or may not be highly symmetric. Moreover, for many harmonic frames of M vectors the symmetry group has order M (cf. [10]), which implies that they are not highly symmetric.

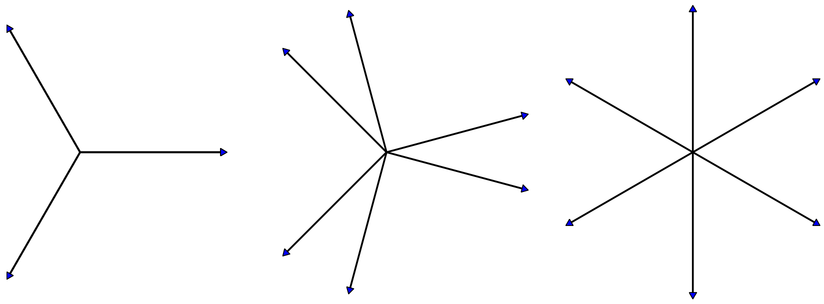


Fig. 4 Unitarily inequivalent tight D_3–frames for $\mathbb{R}^2$ given by the orbit of a vector v.

Example 16. The vertices of the platonic solids in $\mathbb{R}^3$, and the M equally spaced unit vectors in $\mathbb{R}^2$ are highly symmetric tight frames.

Theorem 10. *Fix $M \geq N$. There is a finite number of highly symmetric Parseval frames of M vectors for $\mathbb{F}^N$ (up to unitary equivalence).*

Proof. Suppose that Φ is a highly symmetric Parseval frame of M vectors for $\mathbb{F}^N$. Then it is determined, up to unitary equivalence, by the representation induced by $\mathrm{Sym}(\Phi)$, and a subgroup H which fixes only the one–dimesional subspace spanned by some vector in Φ. There is a finite number of choices for $\mathrm{Sym}(\Phi)$ since its order is $\leq |S_M| = M!$, and hence (by Maschke's theorem) a finite number of possible representations. As there is only a finite number of choices for H, it follows that the class of such frames is finite. □

The highly symmetric tight frames have only recently been defined in [4], where those corresponding to the Shephard–Todd classification of the *finite reflection groups* and *complex polytopes* were enumerated. We give a couple of examples ([4]):

Example 17. Let $G = G(1,1,8) \cong S_8$, a member of one of the three infinite families of *imprimitive irreducible complex reflection groups* acting as permutations of the indices of a vector $x \in \mathbb{C}^8$ in the subspace consisting of vectors with $x_1 + \cdots + x_8 = 0$. The orbit of the vector

$$v = 3w_2 = (3,3,-1,-1,-1,-1,-1,-1).$$

gives an equiangular tight frame of 28 vectors for a 7–dimensional space.

Example 18. The *Hessian* is the regular complex polytope with 27 vertices and Schläfli symbol $3\{3\}3\{3\}3$. Its symmetry group (Shephard–Todd) ST 25 (of order 648) is generated by the following three reflections of order three

$$R_1 = \begin{pmatrix} \omega & & \\ & 1 & \\ & & 1 \end{pmatrix}, \quad R_2 = \frac{1}{3}\begin{pmatrix} \omega+2 & \omega-1 & \omega-1 \\ \omega-1 & \omega+2 & \omega-1 \\ \omega-1 & \omega-1 & \omega+2 \end{pmatrix}, \quad R_3 = \begin{pmatrix} 1 & & \\ & 1 & \\ & & \omega \end{pmatrix}, \quad \omega = e^{\frac{2\pi i}{3}},$$

and it has $v=(1,-1,0)$ as a vertex (cf. [6]). These vertices are the H–orbit of v, with H the Heisenberg group, which is a *Heisenberg frame* (see §9). In particular, they are a highly symmetric tight frame. We observe that H is normal in $G=\langle R_1,R_2,R_3\rangle$.

The classification of all highly symmetric tight frames is in its infancy.

8 Central G–frames

To narrow down the class of unitarily inequivalent G–frames for G nonabelian (which is infinite), we impose an additional symmetry condition:

Definition 10. A G–frame $\Phi=(\varphi_g)_{g\in G}$ is said to be **central** if $\nu: G\to\mathbb{C}$ defined by

$$\nu(g):=\langle\varphi_1,\varphi_g\rangle=\langle\varphi_1,g\varphi_1\rangle$$

is a class function, i.e., is constant on the conjugacy classes of G.

It is easy to see being central is equivalent to the *symmetry condition*

$$\langle g\varphi,h\varphi\rangle=\langle g\psi,h\psi\rangle, \qquad \forall g,h\in G,\ \forall\varphi,\psi\in\Phi.$$

Example 19. For G abelian, all G–frames are central, since the conjugacy classes of an abelian group are singletons.

Thus central G–frames generalise harmonic frames to G nonabelian.

Definition 11. Let $\rho: G\to\mathscr{U}(\mathscr{H})$ be a representation of a finite group G. The **character** of ρ is the map $\chi=\chi_\rho: G\to\mathbb{C}$ defined by

$$\chi(g):=\operatorname{trace}(\rho(g)).$$

We now characterise all central Parseval G–frames in terms of the Gramian. In particular, it turns out that the class of central G–frames is *finite*.

Theorem 11. *([18]) Let G be a finite group with irreducible characters $\chi_1,\ldots,\chi_r$. Then $\Phi=(\varphi_g)_{g\in G}$ is a central Parseval G–frame if and only if its Gramian is given by*

$$\operatorname{Gram}(\Phi)_{g,h}=\sum_{i\in I}\frac{\chi_i(1)}{|G|}\overline{\chi_i}(g^{-1}h), \tag{5}$$

for some $I\subset\{1,\ldots,r\}$.

The central G–frames can be constructed from the irreducible characters of G, in a similar way to the harmonic frames.

Corollary 3. *Let G be a finite group with irreducible characters $\chi_1,\ldots,\chi_r$. Choose Parseval G–frames Φ_i for $\mathscr{H}_i$, $i=1,\ldots,r$, with*

$$\mathrm{Gram}(\Phi_i) = \frac{\chi_i(1)}{|G|} M(\overline{\chi_i}), \qquad \dim(\mathscr{H}_i) = \chi_i(1)^2,$$

e.g., take the columns of $\mathrm{Gram}(\Phi_i)$. *Then the unique (up to unitary equivalence) central Parseval G–frame with Gramian* (5) *is given by the direct sum*

$$\oplus_{i\in I}\Phi_i \subset \mathscr{H} := \oplus_{i\in I}\mathscr{H}_i.$$

Further, if $\rho_i : G \to U(\mathbb{C}^{d_i})$ *is a representation with character* χ_i*, then* Φ_i *can be given as*

$$\Phi_i := \sqrt{\frac{\chi_i(1)}{|G|}} (\rho_i(g))_{g\in G} \subset U(\mathbb{C}^{d_i}) \subset \mathbb{C}^{d_i\times d_i} \approx \mathbb{C}^{d_i^2}, \tag{6}$$

where the inner product on the space of $d_i \times d_i$ *matrices is* $\langle A, B\rangle := \mathrm{trace}(B^*A)$.

Example 20. Let $G = D_3 \cong S_3$ be the dihedral group (symmetric group) of order 6

$$G = D_3 = \langle a, b : a^3 = 1, b^2 = 1, b^{-1}ab = a^{-1}\rangle,$$

and write class functions and G–matrices with respect to the order $1, a, a^2, b, ab, a^2b$. The conjugacy classes are $\{1\}, \{a, a^2\}, \{b, ab, a^2b\}$, and the irreducible characters are

$$\chi_1 = \begin{bmatrix}1\\1\\1\\1\\1\\1\end{bmatrix}, \quad \chi_2 = \begin{bmatrix}1\\1\\1\\-1\\-1\\-1\end{bmatrix}, \qquad \chi_3 = \begin{bmatrix}2\\-1\\-1\\0\\0\\0\end{bmatrix}.$$

Corresponding to each of these, there is a central Parseval G–frame Φ_i for a space of dimension $\chi_i(1)^2$. Since χ_1 and χ_2 are 1–dimensional, (6) gives

$$\Phi_1 = \frac{1}{\sqrt{6}}(1,1,1,1,1,1), \qquad \Phi_2 = \frac{1}{\sqrt{6}}(1,1,1,-1,-1,-1).$$

A representation $\rho : D_3 \to U(\mathbb{C}^2) \subset \mathbb{C}^{2\times 2} \approx \mathbb{C}^4$ with $\mathrm{trace}(\rho) = \chi_3$ is given by

$$\rho(1) = \begin{pmatrix}1&0\\0&1\end{pmatrix} \approx \begin{bmatrix}1\\0\\0\\1\end{bmatrix}, \quad \rho(a) = \begin{pmatrix}\omega&0\\0&\omega^2\end{pmatrix} \approx \begin{bmatrix}\omega\\0\\0\\\omega^2\end{bmatrix}, \quad \rho(a^2) = \begin{pmatrix}\omega^2&0\\0&\omega\end{pmatrix} \approx \begin{bmatrix}\omega^2\\0\\0\\\omega\end{bmatrix},$$

$$\rho(b) = \begin{pmatrix}0&1\\1&0\end{pmatrix} \approx \begin{bmatrix}0\\1\\1\\0\end{bmatrix}, \quad \rho(ab) = \begin{pmatrix}0&\omega\\\omega^2&0\end{pmatrix} \approx \begin{bmatrix}0\\\omega\\\omega^2\\0\end{bmatrix}, \quad \rho(a^2b) = \begin{pmatrix}0&\omega^2\\\omega&0\end{pmatrix} \approx \begin{bmatrix}0\\\omega^2\\\omega\\0\end{bmatrix},$$

and so we obtain from (6)

$$\Phi_3 = \frac{1}{\sqrt{3}}\Big(\begin{bmatrix}1\\0\\0\\1\end{bmatrix}, \begin{bmatrix}\omega\\0\\0\\\omega^2\end{bmatrix}, \begin{bmatrix}\omega^2\\0\\0\\\omega\end{bmatrix}, \begin{bmatrix}0\\1\\1\\0\end{bmatrix}, \begin{bmatrix}0\\\omega\\\omega^2\\0\end{bmatrix}, \begin{bmatrix}0\\\omega^2\\\omega\\0\end{bmatrix}\Big).$$

Thus there are seven central Parseval D_3–frames, namely

$$\Phi_1, \Phi_2 \subset \mathbb{C}, \qquad \Phi_1 \oplus \Phi_2 \subset \mathbb{C}^2, \qquad \Phi_3 \subset \mathbb{C}^4$$

$$\Phi_1 \oplus \Phi_3, \Phi_2 \oplus \Phi_3 \subset \mathbb{C}^5, \qquad \Phi_1 \oplus \Phi_2 \oplus \Phi_3 \subset \mathbb{C}^6.$$

9 Heisenberg frames (SIC-POVMs) Zauner's conjecture.

The Mercedes–Benz frame gives three equiangular lines in $\mathbb{R}^2$. The search for such sets of equiangular lines in $\mathbb{R}^N$ has a long history, and effectively spawned the area of *algebraic graph theory* (see [7]).

Recently, sets of $M = N^2$ equiangular lines in $\mathbb{C}^N$, equivalently equiangular tight frames of $M = N^2$ vectors in $\mathbb{C}^N$, have been constructed numerically, and, in some cases, analytically. We note that N^2 is the maximum number of vectors possible for an equiangular tight frame for $\mathbb{C}^N$ ([15]). Such frames are known as **SIC-POVMs** (symmetric informationally complete positive operator valued measures) in quantum information theory (see [15]), where they are of considerable interest. The claim that they exist for all N is usually known as **Zauner's conjecture** (see [22]).

We now explain how such equiangular tight frames have been, and are expected to be constructed – as the orbit a (Heisenberg) group.

Fix $N \geq 1$, and let ω be the primitive N–th root of unity

$$\omega := e^{2\pi i/N}.$$

Let $T \in \mathbb{C}^{N\times N}$ be the cyclic shift matrix, and $\Omega \in \mathbb{C}^{N\times N}$ the diagonal matrix

$$T := \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ . & & . & & & . \\ . & & & . & & . \\ 0 & 0 & 0 & & 1 & 0 \end{bmatrix}, \qquad \Omega := \begin{bmatrix} 1 & 0 & 0 & \cdot\cdot & 0 \\ 0 & \omega & 0 & \cdot\cdot & 0 \\ 0 & 0 & \omega^2 & & 0 \\ . & . & & . & \\ . & . & & & . \\ 0 & 0 & 0 & & \omega^{N-1} \end{bmatrix}.$$

These have order N, i.e., $T^N = \Omega^N = Id$, and satisfy the **commutativity relation**

$$\Omega^k T^j = \omega^{jk} T^j \Omega^k. \tag{7}$$

In particular, the group generated by T and Ω contains the scalar matrices $\omega^r Id$.

Definition 12. The group $H = \langle T, \Omega \rangle$ generated by the matrices T and Ω is called the **discrete Heisenberg group modulo** N, or for short the **Heisenberg group**.

In view of (7), the Heisenberg group has order N^3, and is given explicitly by

$$H = \{\omega^r T^j \Omega^k : 0 \leq r, j, k \leq N-1\}.$$

Since ω, T, Ω have order N, it is convenient to allow the indices of $\omega^r T^j \Omega^k$ to be integers modulo N. Since T and Ω are unitary, H is a group of unitary matrices.

The action of H on $\mathbb{C}^N$ is irreducible, and so by Theorem 3, every orbit $(gv)_{g \in H}$, $v \neq 0$ is a tight frame for $\mathbb{C}^N$. For j, k fixed, the N vectors $\omega^r T^j \Omega^k v$, $0 \leq r \leq N-1$ are scalar multiples of each other, which we identify together. It is in this sense that the orbit of H is interpreted as a set of N^2 (hopefully equiangular) vectors:

$$\Phi := \{T^j \Omega^k v\}_{(j,k) \in \mathbb{Z}_N \times \mathbb{Z}_N}. \tag{8}$$

This Φ is the *Gabor system* given by the subset $\Lambda = \mathbb{Z}_N \times \mathbb{Z}_N \cong G \times \hat{G}$, $G = \mathbb{Z}_N$ (see Chapter X – Gabor frames).

Definition 13. We call a tight frame Φ of the form (8) a **Heisenberg frame** if it is an equiangular tight frame, i.e., a SIC–POVM, and the v a **generating vector**.

Example 21. The vector

$$v = \frac{1}{\sqrt{6}} \begin{pmatrix} \sqrt{3+\sqrt{3}} \\ e^{\frac{\pi}{4} i} \sqrt{3-\sqrt{3}} \end{pmatrix}$$

generates a Heisenberg frame of 4 equiangular vectors for $\mathbb{C}^2$. To date (see [16]), there are known analytic solutions for $N = 2, 3, \ldots, 15, 19, 24, 35, 48$.

Starting with [15], there have been numerous attempts to find generating vectors v for various dimensions N, starting from numerical solutions. The current state of affairs is summarised in [16]. We now outline some of the salient points.

The key ideas for finding generating vectors are:

- Solve an equivalent simplified set of equations.
- Find a generating vector with special properties.
- Understand the relationship between generating vectors.

For a unit vector $v \in \mathbb{C}^N$, the condition that it generate a Heisenberg frame is:

$$|\langle gv, hv \rangle| = \frac{1}{\sqrt{N+1}}, \quad j \neq k \quad \Longleftrightarrow \quad |\langle v, T^j \Omega^k v \rangle| = \frac{1}{\sqrt{N+1}}, \quad j, k \in \mathbb{Z}_N.$$

This isn't ammenable to numerical calculation. In [15], the *second frame potential*

$$f(v) = \sum_{j=0}^{N-1} \sum_{k=0}^{N-1} |\langle v, T^j \Omega^k v \rangle|^4,$$

was minimised over all v satisfying $g(v) = \|v\|^2 = 1$. A minimiser of this constained optimisation problem with

$$f(v) = 1 + (N^2 - 1)\frac{1}{(\sqrt{N+1})^4} = \frac{2N}{N+1}$$

is a generating vector. Various simplified equations for finding generators have been proposed, most notably (see [1], [2], [14]):

Theorem 12. *A vector $v = (z_j)_{j\in\mathbb{Z}_N}$ is a generating vector for a Heisenberg frame if and only if*

$$\sum_{j\in\mathbb{Z}_N} z_j \overline{z}_{j+s}\overline{z}_{t+j} z_{j+s+t} = \begin{cases} 0, & s,t \neq 0; \\ \frac{1}{N+1}, & s \neq 0, t = 0, \quad s = 0, t \neq 0; \\ \frac{2}{N+1}, & (s,t) = (0,0). \end{cases}$$

If v generates a Heisenberg frame, b is a unitary matrix which normalises the Heisenberg group, then bv is also a generating vector, since

$$|\langle (bv), g(bv)\rangle| = |\langle v, b^* gbv\rangle| = |\langle v, b^{-1} gbv\rangle| = \frac{1}{\sqrt{N+1}}, \qquad g \in H, g \neq Id.$$

The normaliser of H in the unitary matrices is often called the **Clifford group**. This group contains the Fourier matrix, since

$$F^{-1}(T^j\Omega^k)F = \omega^{-jk}T^k\Omega^{-j} \in H.$$

and the matrix Z given by

$$(Z)_{jk} := \frac{1}{\sqrt{d}}\mu^{j(j+d)+2jk}, \qquad \mu := e^{\frac{2\pi i}{2N}} = \omega^{\frac{1}{2}},$$

since

$$Z^{-1}(T^j\Omega^k)Z = \mu^{j(d+j-2k)}T^{k-j}\Omega^{-j}.$$

A scalar multiple of Z has order 3, i.e., $Z^3 = \sqrt{i}^{1-d}$, $\sqrt{i} := e^{\frac{2\pi i}{8}}$. The strong form of Zauner's conjecture is:

Conjecture 1. (Zauner). Every generating vector for a Heisenberg frame (up to unitary equivalence) is an eigenvecter of Z.

All known generating vectors (both numerical and analytic) support this conjecture. Indeed, many were found as eigenvectors of Z. Without doubt, the solution of Zauner's conjecture, and the construction of equiangular tight frames in general, is one of the central problems in the construction of tight frames via groups. This field in still in its infancy: frames given as the orbit of more than one vector (G–invariant fusion frames) have scarcely been studied.

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