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Orthogonal polynomials on the disc

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Abstract

We consider the space $\mathcal{P}_n$ of orthogonal polynomials of degree n on the unit disc for a general radially symmetric weight function. We show that there exists a single orthogonal polynomial whose rotations through the angles $\frac{j\pi}{n+1}$, $j = 0, 1, \ldots, n$ forms an orthonormal basis for $\mathcal{P}_n$, and compute all such polynomials explicitly. This generalises the orthonormal basis of Logan and Shepp for the Legendre polynomials on the disc.

Furthermore, such a polynomial reflects the rotational symmetry of the weight in a deeper way: its rotations under other subgroups of the group of rotations forms a tight frame for $\mathcal{P}_n$, with a continuous version also holding. Along the way, we show that other frame decompositions with natural symmetries exist, and consider a number of structural properties of $\mathcal{P}_n$ including the form of the monomial orthogonal polynomials, and whether or not $\mathcal{P}_n$ contains ridge functions.

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1. Introduction

Here we consider the space $\mathcal{P}_n = \mathcal{P}_n^w$ of orthogonal polynomials of degree n on the unit disc

$$\mathbb{D} := \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leqslant 1\}$$

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for a suitable radially symmetric weight function given by $w : [0, 1] \to \mathbb{R}^+$ (or more generally a measure). This $n+1$ dimensional space consists of all polynomials of degree n which are orthogonal to all polynomials of degree $< n$ with respect to the corresponding inner product

$$\langle f, g\rangle = \langle f, g\rangle_w := \int_{\mathbb{D}} fgw = \int_0^{2\pi}\int_0^1 (fg)(r\cos\theta, r\sin\theta)w(r)r\,dr\,d\theta. \tag{1.1}$$

We are primarily interested in the *Gegenbauer polynomials*, which are given by the weight

$$w(r) := (1-r^2)^\alpha, \quad \alpha > -1. \tag{1.2}$$

These polynomials have long been used to analyse the optical properties of a circular lens, and to reconstruct images from Radon projections, see, e.g., [5,6].

Let $R_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ denote rotation through the angle θ, i.e.,

$$R_\theta(x, y) := \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta \end{pmatrix}.$$

Let the group of rotations of the disc (which are symmetries of the weight)

$$\mathrm{SO}(2) = \{R_\theta : 0 \leqslant \theta < 2\pi\}$$

act on functions defined on the disc in the natural way, i.e.,

$$R_\theta f := f \circ R_\theta^{-1}.$$

Logan and Shepp [3] showed that the *Legendre polynomials* on the disc (constant weight $w = 1$) have an orthonormal basis given by the $n+1$ polynomials

$$p_j(x, y) := \frac{1}{\sqrt{\pi}} U_n\left(x\cos\frac{j\pi}{n+1} + y\sin\frac{j\pi}{n+1}\right), \quad j = 0, \ldots, n, \tag{1.3}$$

where U_n is the nth *Chebyshev polynomial of the second kind*. This result says that an orthonormal basis can be constructed from a single simple polynomial p_0 (a ridge function obtained from a univariate orthogonal polynomial) by rotating it through the angles $\frac{j\pi}{n+1}$, $0 \leqslant j \leqslant n$. In this paper we explore how this can be extended for a general radially symmetric weight. It turns out that such an orthogonal expansion always exists, though the 'simple' polynomial p_0 is not in general a ridge function. Moreover, such an expansion reflects the rotational symmetry of the weight in a deeper way, e.g., for Legendre polynomials there exists the so-called tight frame decompositions

$$f = \frac{n+1}{k}\sum_{j=0}^{k-1}\langle f, R^j_{\frac{2\pi}{k}} p_0\rangle R^j_{\frac{2\pi}{k}} p_0 = \frac{n+1}{2\pi}\int_0^{2\pi}\langle f, R_\theta p_0\rangle R_\theta p_0\,d\theta, \quad \forall f \in \mathcal{P}_n,$$

where p_0 is given by (1.3), and $k \geqslant n+1$ with k not even if $k \leqslant 2n$.

The paper is set out as follows. In the remainder of this section we give formulas for the inner product, and discuss ridge functions and Zernike polynomials. In Section 2, we discuss symmetries of tight frames as they apply to $\mathcal{P}_n$. We show that orthogonal and biorthogonal systems with rotational symmetries always exist, and that the corresponding expansions automatically inherit a higher degree of rotational symmetry than would be expected. In Section 3, we use the orthogonal decomposition of $\mathcal{P}_n$ into SO(2)-invariant subspaces to find an explicit formula for all polynomials $p \in \mathcal{P}_n$ for which $\left\{R^j_{\frac{\pi}{n+1}} p\right\}_{j=0}^n$ is an orthonormal basis for $\mathcal{P}_n$.

1.1. The inner product

It is convenient to allow the orthogonal polynomials in $\mathcal{P}_n$ to have complex coefficients, and at times replace the cartesian coordinates x and y by z and $\overline{z}$, where $z := x + iy$. We also allow the formula for a polynomial (in either system) to appear in place of the function in the inner product and the integral defining it, e.g., by integrating the polar form, we have

$$\langle z^j \overline{z}^k, 1\rangle = \int_{\mathbb{D}} z^j \overline{z}^k w(|z|) = \begin{cases} 0, & j \neq k, \\ m_j, & j = k, \end{cases} \tag{1.4}$$

where

$$m_j := \int_{\mathbb{D}} |z|^{2j} w(|z|) = 2\pi \int_0^1 r^{2j+1} w(r)\, dr > 0, \quad j = 0, 1, \ldots .$$

By symmetry $\langle x^{j_1} y^{k_1}, x^{j_2} y^{k_2}\rangle = 0$ unless $j_1 + j_2$ and $k_1 + k_2$ are both even, in which case the inner product is given by

$$\langle x^{2j} y^{2k}, 1\rangle = I(j,k) m_{j+k}, \quad j, k \geqslant 0, \tag{1.5}$$

where

$$I(j,k) := \frac{1}{2\pi} \int_0^{2\pi} \cos^{2j}\theta \sin^{2k}\theta \, d\theta = \frac{1 \cdot 3 \cdots (2j-1) \cdot 1 \cdot 3 \cdots (2j-1)}{2 \cdot 4 \cdots (2j+2k)}.$$

For example, the inner products of quintic polynomials can be computed using

$$\langle 1, 1\rangle = m_0, \quad \langle x^2, 1\rangle = \langle y^2, 1\rangle = \tfrac{1}{2} m_1, \quad \langle x^4, 1\rangle = \langle y^4, 1\rangle = \tfrac{3}{8} m_2,$$
$$\langle x^2 y^2, 1\rangle = \tfrac{1}{8} m_2. \tag{1.6}$$

For the Gegenbauer weight (1.2), the 'moments' m_j are given by

$$m_j = \frac{j!\pi}{(\alpha+1)_{j+1}}, \quad j = 0, 1, \ldots . \tag{1.7}$$

By the Cauchy–Schwarz inequality

$$m_{k-2} m_k - m_{k-1}^2 > 0, \quad k \geqslant 2. \tag{1.8}$$

The values (1.8) appear in the denominators of some of the formulas which follow.

1.2. Ridge functions

A nonzero function f on the disc is called *ridge function* (or *plane wave*) if it can be written as a univariate map $g : [-1,1] \to \mathbb{R}$ composed with a linear map $\langle \cdot, v\rangle : \mathbb{R}^2 \to \mathbb{R}$, $v \in \mathbb{R}^2$, $\|v\| = 1$, i.e.,

$$f(x,y) = g(\langle \begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \rangle) = g(v_1 x + v_2 y), \quad \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{D}.$$

In particular, if $v = e_1 = (1,0)$ the first standard basis vector, then $f(x,y) = g(x)$.

There exist ridge functions in $\mathcal{P}_n$, as in the case of the Legendre polynomials, if and only if the orthogonal projection of $\langle\cdot, e_1\rangle^n : (x, y) \mapsto x^n$ onto $\mathcal{P}_n$ is a ridge function, i.e., is a function only of x. This may or may not be the case.

Example 1. The Gegenbauer polynomials contain ridge polynomials (this is also true for a ball in any number of dimensions (cf. [2, Proposition 6.1.13])), namely

$$P_n^{(\alpha+\frac{1}{2},\alpha+\frac{1}{2})}(\langle\cdot, v\rangle), \quad \|v\| = 1,$$

where

$$\|P_k^{(\alpha+\frac{1}{2},\alpha+\frac{1}{2})}(\langle\cdot, v\rangle)\|^2 = \frac{2^{2\alpha+1}\Gamma(\alpha+1)^2}{\Gamma(2\alpha+2)} \frac{2^{2\alpha+2}\Gamma(n+\alpha+\frac{3}{2})^2}{n!(2n+\alpha+2)\Gamma(n+2\alpha+2)}.$$

For a general radially symmetric weight the orthogonal projections of $(x, y) \mapsto x^n$ onto $\mathcal{P}_n$ for the first few n are given by

$$1, \quad x, \quad x^2 - \frac{m_1}{2m_0}, \quad x^3 - \frac{3m_2}{4m_1}x,$$

which are ridge functions. For $n = 4$, the formula is

$$x^4 - \frac{3m_1m_2^2 + 4m_1^2m_3 - 7m_0m_2m_3}{8m_2(m_1^2 - m_0m_2)}x^2 + \frac{4m_1^2m_3 - 3m_1m_2^2 - m_0m_2m_3}{8m_2(m_1^2 - m_0m_2)}y^2$$
$$+\frac{3}{8}\frac{m_2^2 - m_1m_3}{m_1^2 - m_0m_2},$$

which is not a ridge function if

$$4m_1^2m_3 - 3m_1m_2^2 - m_0m_2m_3 \neq 0.$$

Let w be the radially symmetric weight given by

$$w(r) := r^{2\beta}, \quad \beta > -1, \quad m_j = \frac{\pi}{j+1+\beta}. \tag{1.9}$$

Then the orthogonal projection of $(x, y) \mapsto x^4$ onto $\mathcal{P}_4$ is given by

$$x^4 - \frac{1}{4}\frac{5\beta+12}{\beta+4}x^2 - \frac{1}{4}\frac{\beta}{\beta+4}y^2 + \frac{3}{8}\frac{\beta^2+3\beta+2}{\beta^2+7\beta+12},$$

which is a ridge function if only if $\beta = 0$. Hence for the inner product given by (1.9) with $\beta \neq 0$, $\mathcal{P}_4$ does not contain any ridge functions.

As the above example indicates, the orthogonal projection of $(x, y) \mapsto x^j y^{n-j}$ onto $\mathcal{P}_n$ is even if n is even, and odd if n is odd. Moreover, for the Gegenbauer polynomials, only the powers $x^{\beta_1}y^{\beta_2}$ with $(\beta_1, \beta_2) \leqslant (j, n-j)$ have nonzero coefficients.

1.3. Zernike polynomials

From (1.4) it follows that the orthogonal projections of $z \mapsto z^j\overline{z}^k$, $j + k = n$ onto $\mathcal{P}_n$ form an orthogonal basis for $\mathcal{P}_n$. For the Gegenbauer weight these polynomials are given by the formula

$$P^{\alpha}_{j,k}(z) := \frac{(\alpha+1)_{j+k}}{(\alpha+1)_j(\alpha+1)_k} z^j \overline{z}^k {}_2F_1\left(\begin{matrix} -j,-k \\ -\alpha-j-k \end{matrix}; \frac{1}{z\overline{z}}\right)$$
$$= \frac{(\alpha+1)_{j+k}}{(\alpha+1)_j(\alpha+1)_k} z^j \overline{z}^k + \text{lower order terms},$$

and have the factorisation (cf. [2, Section 2.4.3])

$$P^{\alpha}_{n-j,j}(z) = \frac{j!}{(\alpha+1)_j} z^{n-2j} P_j^{(\alpha,n-2j)}(2|z|^2-1), \quad n-j \geqslant j. \tag{1.10}$$

These polynomials are often referred to as *Zernike polynomials* or *disc polynomials*, see, e.g., [2,5]. The Zernike polynomials for a general radially symmetric weight satisfy a factorisation similar to (1.10).

Lemma 1.11. *Fix a weight function* $w : [0,1] \to \mathbb{R}^+$. *Let* $0 \leqslant j \leqslant \frac{n}{2}$, *and* $P_j \neq 0$ *be an orthogonal polynomial of degree j for the univariate weight* $(1+x)^{n-2j} w\left(\sqrt{\frac{1+x}{2}}\right)$ *on* $[-1,1]$. *Then the polynomials of degree n given by the formulas*

$$Q^w_{j,n}(z) := z^{n-2j} P_j(2|z|^2-1), \quad Q^w_{j,n}(\overline{z}) = \overline{z}^{n-2j} P_j(2|z|^2-1), \tag{1.12}$$

belong to $\mathcal{P}_n = \mathcal{P}^w_n$. *Moreover the set of these polynomials are an orthogonal basis for* $\mathcal{P}_n$, *with their norms given by*

$$h_j := \||z|^{n-2j} P_j(2|z|^2-1)\|^2 = \frac{\pi}{2^{n-2j+1}} \int_{-1}^{1} P_j^2(x)(1+x)^{n-2j} w\left(\sqrt{\frac{1+x}{2}}\right) dx. \tag{1.13}$$

Proof. If $p \in \mathcal{P}_n$, then so is the polynomial $z \mapsto \overline{p(z)}$, and so it suffices to show that the first of these polynomials is in $\mathcal{P}_n$. This polynomial has the form

$$Q^w_{j,n}(z) = z^{n-2j} P_j(2|z|^2-1) = \sum_{k=0}^{j} c_k z^{n-2j+k} \overline{z}^k,$$

and so, by (1.4), is orthogonal to all monomials of degree $< n$, except possibly

$$z \mapsto \overline{z}^{n-2j} z^s \overline{z}^s, \quad s = 0,1,\ldots,j-1.$$

By making the change of variables $x = 2r^2 - 1$, the condition for orthogonality to these can be written

$$\langle Q^w_{j,n}(z), \overline{z}^{n-2j} z^s \overline{z}^s \rangle = \int_{\mathbb{D}} P_j(2|z|^2-1)|z|^{2s}|z|^{2n-4j} w(|z|)$$
$$= 2\pi \int_0^1 P_j(2r^2-1) r^{2s} r^{2n-4j} w(r) r \, dr$$
$$= 2\pi \int_{-1}^{1} P_j(x) \left(\frac{1+x}{2}\right)^s \left(\frac{1+x}{2}\right)^{n-2j} w\left(\sqrt{\frac{1+x}{2}}\right) \frac{dx}{4} = 0$$

for $0 \leqslant s < j$, which is satisfied by the choice of P_j. Similarly, we compute

$$h_j := \||z|^{n-2j} P_j(2|z|^2-1)\|^2 = \|Q^w_{j,n}(z)\|^2 = \|Q^w_{j,n}(\overline{z})\|^2$$
$$= 2\pi \int_0^1 P_j^2(2r^2-1) r^{2n-4j} w(r) r \, dr$$
$$= 2\pi \int_{-1}^1 P_j^2(x) \left(\frac{1+x}{2}\right)^{n-2j} w\left(\sqrt{\frac{1+x}{2}}\right) \frac{dx}{4}. \qquad \square$$

Example 1. The first two monic polynomials are given by

$$P_0(x) = 1, \quad P_1(x) = x + \frac{m_{n-2} - 2m_{n-1}}{m_{n-2}}, \quad n \geqslant 2.$$

Thus using (1.13) and (1.4), we obtain

$$\frac{1}{\sqrt{h_0}} P_0(2|z|^2-1) = \frac{1}{\sqrt{m_n}}, \quad n \geqslant 0, \tag{1.14}$$

$$\frac{1}{\sqrt{h_1}} P_1(2|z|^2-1) = \frac{\sqrt{m_{n-2}}}{\sqrt{m_{n-2}m_n - m_{n-1}^2}} \left(|z|^2 - \frac{m_{n-1}}{m_{n-2}}\right), \quad n \geqslant 2. \tag{1.15}$$

Example 2. If the radial weight is given by the generalised Gegenbauer weight

$$w(r) := (1-r^2)^\alpha r^{2\beta}, \quad \alpha > -1, \quad \beta > -1,$$

then

$$(1+x)^{n-2j} w\left(\sqrt{\frac{1+x}{2}}\right) = (1+x)^{n-2j} \left(\frac{1-x}{2}\right)^\alpha \left(\frac{1+x}{2}\right)^\beta$$
$$= \frac{1}{2^{\alpha+\beta}} (1-x)^\alpha (1+x)^{n-2j+\beta},$$

so that P_j is the Jacobi polynomial $P_j^{(\alpha,n-2j+\beta)}$, for which

$$h_j = \frac{\pi}{2^{n-2j+1}} \frac{1}{2^{\alpha+\beta}} \int_{-1}^1 (P_j^{(\alpha,n-2j+\beta)}(x))^2 (1-x)^\alpha (1+x)^{n-2j+\beta} \, dx$$
$$= \pi \frac{\Gamma(j+\alpha+1)\Gamma(n-j+\beta+1)}{(\alpha+n+\beta+1) j! \Gamma(\alpha+n-j+\beta+1)}. \tag{1.16}$$

2. Frames and their symmetries

We outline the basics of finite frame theory (cf. [1,4]) as they apply to our construction. Given a finite spanning set $\Phi = \{\phi_j\}$ (called a *frame*) for a finite dimensional Hilbert space $\mathcal{H}$, such as $\mathcal{P}_n$, the self adjoint operator $S = S_\Phi : \mathcal{H} \to \mathcal{H}$ given by

$$Sf := \sum_j \langle f, \phi_j \rangle \phi_j, \quad \forall f \in \mathcal{H} \tag{2.1}$$

is positive and invertible. The set $\{\tilde{\phi}_j\}$, $\tilde{\phi}_j := S^{-1}\phi_j$ is called the *dual frame*, and gives the expansion

$$f = \sum_j \langle f, \phi_j \rangle \tilde{\phi}_j = \sum_j \langle f, \tilde{\phi}_j \rangle \phi_j, \qquad \forall f \in \mathcal{H}.$$

Special cases include orthogonal and biorthogonal expansions. Moreover, if $\psi_j := S^{-\frac{1}{2}}\phi_j$, then we have the *canonical tight frame* decomposition

$$f = \sum_j \langle f, \psi_j \rangle \psi_j, \quad \forall f \in \mathcal{H}.$$

Suppose that G is group of unitary transformations of $\mathcal{H}$ which maps Φ to itself. Then each $g \in G$ commutes with S

$$g(Sf) = \sum_j \langle f, \phi_j \rangle g\phi_j = \sum_j \langle f, g^{-1}g\phi_j \rangle g\phi_j = \sum_j \langle gf, g\phi_j \rangle g\phi_j = S(gf),$$

and hence with S^{-1} and $S^{-\frac{1}{2}}$. Thus if the G-orbit of a vector ϕ_0 spans $\mathcal{H}$, then the corresponding dual and canonical dual frames are the G-orbit of a vector. This result allows us to take a spanning set for $\mathcal{P}_n$ given by the rotates of a single polynomial (which is easy to find) and convert it into a tight frame which is given by the corresponding rotates of a single polynomial. We now illustrate this.

Any polynomial $p \in \mathcal{P}_n$ is a multiple of its rotation through π

$$R_\pi p = (-1)^n p, \quad \forall p \in \mathcal{P}_n, \tag{2.2}$$

and so for the rotations of p by multiples of $\frac{2\pi}{k}$ to span the $n+1$ dimensional space $\mathcal{P}_n$, we must have either $k \geqslant n+1$ and k odd, or $k \geqslant 2(n+1)$. Under these conditions we can find such a polynomial p.

Lemma 2.3. *Let ϕ be the orthogonal projection of $(x, y) \mapsto x^n$ onto $\mathcal{P}_n$. If either $k \geqslant n+1$ and k is odd, or $k \geqslant 2(n+1)$, then the rotations of ϕ through multiples of $\frac{2\pi}{k}$, i.e., the set*

$$\Phi = \{\phi_j\}_{j=0}^{k-1}, \quad \phi_j := R^j\phi, \quad R := R_{\frac{2\pi}{k}},$$

spans $\mathcal{P}_n$. Thus, there exists $\phi, \tilde{\phi}, \psi \in \mathcal{P}_n$ for which

$$f = \sum_{j=0}^{k-1} \langle f, R^j\phi \rangle R^j\tilde{\phi} = \sum_{j=0}^{k-1} \langle f, R^j\tilde{\phi} \rangle R^j\phi = \sum_{j=0}^{k-1} \langle f, R^j\psi \rangle R^j\psi, \quad \forall f \in \mathcal{P}_n. \tag{2.4}$$

In particular, by taking $k = 2(n+1)$, we conclude that there exists a polynomial $v \in \mathcal{P}_n$ for which $\left\{R^j_{\frac{\pi}{n+1}} v\right\}_{j=0}^{n}$ is an orthonormal basis.

Proof. In view of (2.2), we may assume without loss of generality that $k \geqslant 2(n+1)$. Since an orthogonal polynomial in $\mathcal{P}_n$ is uniquely determined by its leading term

$$\phi(x, y) = \phi_\downarrow(x, y) + \text{lower order terms}, \quad \phi_\downarrow(x, y) := x^n,$$

and

$$(R^j \phi)(x, y) = (R^j \phi_\downarrow)(x, y) + \text{lower order terms}.$$

Hence to show Φ spans $\mathcal{P}_n$, it suffices to show that the rotations of $\phi_\downarrow : (x, y) \mapsto x^n$ through the angles $\frac{2\pi j}{k}$, $0 \leqslant j \leqslant n$ are linearly independent, i.e., using the complex notation, that

$$\sum_{j=0}^{n} c_j (\omega^j z + \overline{\omega^j z})^n = \sum_{j=0}^{n} c_j \sum_{r=0}^{n} \binom{n}{r} (\omega^j z)^r (\overline{\omega^j z})^{n-r} = 0, \quad \omega := e^{\frac{2\pi i}{k}}$$

implies all the coefficients c_j are zero. From the orthogonality between $z^r \overline{z}^{n-r}$, $0 \leqslant r \leqslant n$ given by (1.4), we obtain

$$\sum_{j=0}^{n} c_j (\omega^{2r-n})^j = 0, \quad 0 \leqslant r \leqslant n.$$

Thus the polynomial $z \mapsto \sum_{j=0}^{n} c_j z^n$ has $n+1$ distinct roots ω^{2r-n}, $0 \leqslant r \leqslant n$, and so all of its coefficients c_j are zero.

The decomposition (2.4) follows from the previous discussion, where

$$\tilde{\phi} := S_\Phi^{-1} \phi, \quad \psi := S_\Phi^{-\frac{1}{2}} \phi.$$

Finally, taking $k = 2(n+1)$ and using (2.2), gives

$$f = \sum_{j=0}^{2n+1} \langle f, R^j_{\frac{2\pi}{2(n+1)}} \psi \rangle R^j_{\frac{2\pi}{2(n+1)}} \psi = 2 \sum_{j=0}^{n} \langle f, R^j_{\frac{\pi}{(n+1)}} \psi \rangle R^j_{\frac{\pi}{(n+1)}} \psi, \qquad \forall f \in \mathcal{P}_n,$$

so that $\sqrt{2}\psi$ gives the desired polynomial v. $\square$

If any frame expansion of the type (2.4) holds, then (with appropriate normalisation) it holds for all possible k, including a continuous version.

Theorem 2.5. *If there are polynomials $g, \tilde{g} \in \mathcal{P}_n$ and some $k \in \mathbb{N}$, for which*

$$f = \frac{n+1}{k} \sum_{j=0}^{k-1} \langle f, R^j g \rangle R^j \tilde{g}, \quad \forall f \in \mathcal{P}_n, \quad R := R_{\frac{2\pi}{k}}, \tag{2.6}$$

then (2.6) *holds for any k with either $k \geqslant n+1$ and k odd, or $k \geqslant 2(n+1)$. Moreover,*

$$f = \frac{n+1}{2\pi} \int_0^{2\pi} \langle f, R_\theta g \rangle R_\theta \tilde{g} \, d\theta, \quad \forall f \in \mathcal{P}_n.$$

Proof. Suppose that (2.6) holds for some k. Then $k \geqslant n+1$ (by spanning). Further, we may assume without loss of generality that $k \geqslant 2(n+1)$, since if not then by (2.2) k must be odd, in which case $R_{\frac{\pi}{k}}^{2j+k} = -R_{\frac{2\pi}{k}}^{j}$, giving

$$f = \frac{1}{2}\frac{n+1}{k}\sum_{j=0}^{k-1}\langle f, R_{\frac{\pi}{k}}^{2j} g\rangle R_{\frac{\pi}{k}}^{2j}\tilde{g} + \frac{1}{2}\frac{n+1}{k}\sum_{j=0}^{k-1}\langle f, R_{\frac{\pi}{k}}^{2j+k} g\rangle R_{\frac{\pi}{k}}^{2j+k}\tilde{g}$$

$$= \frac{n+1}{2k}\sum_{j=0}^{2k-1}\langle f, R_{\frac{2\pi}{2k}}^{j} g\rangle R_{\frac{2\pi}{2k}}^{j}\tilde{g}, \quad \forall f \in \mathcal{P}_n.$$

It therefore suffices to prove for $k \geqslant 2(n+1)$ that

$$\frac{1}{k}\sum_{j=0}^{k-1}\langle f, R_{\frac{2\pi}{k}}^{j} g\rangle R_{\frac{2\pi}{k}}^{j}\tilde{g} = \frac{1}{2\pi}\int_0^{2\pi}\langle f, R_\theta g\rangle R_\theta \tilde{g}\, d\theta, \quad \forall f \in \mathcal{P}_n. \tag{2.7}$$

Since $p : \theta \mapsto \langle f, R_\theta g\rangle R_\theta \tilde{g}(x, y)$ is a trigonometric polynomial of degree $2n$, and $k > 2n$, we can integrate it using the quadrature formula for k equally spaced nodes

$$\frac{1}{2\pi}\int_0^{2\pi}\langle f, R_\theta g\rangle R_\theta \tilde{g}(x, y)\, d\theta = \frac{1}{2\pi}\int_0^{2\pi} p(\theta)\, d\theta = \frac{1}{k}\sum_{j=0}^{k-1} p\left(\frac{2\pi j}{k}\right)$$

$$= \frac{1}{k}\sum_{j=0}^{k-1}\langle f, R_{\frac{2\pi j}{k}} g\rangle R_{\frac{2\pi j}{k}}\tilde{g}(x, y), \quad \forall f \in \mathcal{P}_n,$$

thereby obtaining (2.7). □

Let $\mathcal{T}_n$ be the trigonometric polynomials of degree $\leqslant n$, with the usual inner product. If (2.6) holds with $g = \tilde{g}$, then we can naturally associate with each $f \in \mathcal{P}_n$ a trigonometric polynomial $\hat{f} \in \mathcal{T}_n$ given by

$$\hat{f}(\theta) := \langle f, R_\theta g\rangle, \quad f = \frac{n+1}{2\pi}\int_0^{2\pi}\hat{f}(\theta) R_\theta g\, d\theta, \quad \forall f \in \mathcal{P}_n.$$

The map $\mathcal{P}_n \to \mathcal{T}_n : f \mapsto \hat{f}$ is a (complex) Hilbert space isomorphism, since

$$\langle f_1, f_2\rangle = \frac{n+1}{2\pi}\left\langle \int_0^{2\pi}\hat{f}_1(\theta) R_\theta g\, d\theta, f_2\right\rangle = \frac{n+1}{2\pi}\int_0^{2\pi}\hat{f}_1(\theta)\langle R_\theta g, f_2\rangle\, d\theta$$

$$= \frac{n+1}{2\pi}\int_0^{2\pi}\hat{f}_1(\theta)\overline{\hat{f}_2(\theta)}\, d\theta, \quad \forall f_1, f_2 \in \mathcal{P}_n.$$

Example 1. Let $\phi \in \mathcal{P}_2$ be the orthogonal projection of $(x, y) \mapsto x^2$ onto the quadratic orthogonal polynomials $\mathcal{P}_2$, i.e.,

$$\phi(x, y) := x^2 - \frac{m_1}{2m_0}.$$

By Lemma 2.3, $\Phi := \{\phi_0, \phi_1, \phi_2\}$, $\phi_j := R^j_{\frac{2\pi}{3}}\phi$ is a basis for $\mathcal{P}_2$. By (1.6),

$$\langle \phi, R_\theta \phi \rangle = \left\langle x^2 - \frac{m_1}{2m_0}, (x\cos\theta - y\sin\theta)^2 - \frac{m_1}{2m_0} \right\rangle = \frac{1}{4} m_2 \cos^2\theta - \frac{1}{4}\frac{m_1^2}{m_0} + \frac{1}{8} m_2,$$

so the matrix representing S_Φ of (2.1) with respect to this basis is

$$A = \begin{pmatrix} a & b & b \\ b & a & b \\ b & b & a \end{pmatrix}, \quad a := \frac{3}{8} m_2 - \frac{1}{4}\frac{m_1^2}{m_0}, \quad b := \frac{3}{16} m_2 - \frac{1}{4}\frac{m_1^2}{m_0}.$$

This symmetric matrix can be diagonalised

$$A = V \Lambda V^{-1}, \quad V := \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{-2}{\sqrt{6}} & 0 \end{pmatrix}, \quad \Lambda := \begin{pmatrix} \frac{3}{4}\frac{m_0 m_2 - m_1^2}{m_0} & 0 & 0 \\ 0 & \frac{3}{16} m_2 & 0 \\ 0 & 0 & \frac{3}{16} m_2 \end{pmatrix}.$$

From this we calculate

$$\tilde{\phi} = S_\Phi^{-1}\phi = \frac{4}{9}\frac{9m_0m_2 - 8m_1^2}{m_2(m_0m_2 - m_1^2)}\phi - \frac{4}{9}\frac{3m_0m_2 - 4m_1^2}{m_2(m_0m_2 - m_1^2)}(R_{\frac{2\pi}{3}}\phi + R_{\frac{4\pi}{3}}\phi),$$

and $\psi = S_\Phi^{-\frac{1}{2}}\phi$, which gives

$$\psi = \left(\frac{2}{9}\frac{\sqrt{3m_0}}{\sqrt{m_0m_2 - m_1^2}} + \frac{8}{9}\frac{\sqrt{3}}{\sqrt{m_2}} \right)\phi + \left(\frac{2}{9}\frac{\sqrt{3m_0}}{\sqrt{m_0m_2 - m_1^2}} - \frac{4}{9}\frac{\sqrt{3}}{\sqrt{m_2}} \right) \times (R_{\frac{2\pi}{3}}\phi + R_{\frac{4\pi}{3}}\phi).$$

Thus for $k \geqslant 3$, $k \neq 4$, we have

$$f = \frac{3}{k+1}\sum_{j=0}^{k-1}\langle f, R^j_{\frac{2\pi}{k}}\phi\rangle R^j_{\frac{2\pi}{k}}\tilde{\phi} = \frac{3}{k+1}\sum_{j=0}^{k-1}\langle f, R^j_{\frac{2\pi}{k}}\tilde{\phi}\rangle R^j_{\frac{2\pi}{k}}\phi$$

$$= \frac{3}{k+1}\sum_{j=0}^{k-1}\langle f, R^j_{\frac{2\pi}{k}}\psi\rangle R^j_{\frac{2\pi}{k}}\psi, \qquad \forall f \in \mathcal{P}_2.$$

The above computation for ψ is difficult to do for a general n, as it requires the computation of the positive square root of S^{-1}. In the next section, we use a result of [4] to obtain all polynomials $\psi \in \mathcal{P}_n$ whose rotations form an orthonormal basis for $\mathcal{P}_n$, and in particular, that corresponding to the orthogonal projection of $(x, y) \mapsto x^n$ onto $\mathcal{P}_n$.

3. The orthonormal basis

In Lemma 2.3 the existence of a polynomial $v \in \mathcal{P}_n$ for which $\left\{R^j_{\frac{\pi}{n+1}} v\right\}_{j=0}^n$ is an orthonormal basis for $\mathcal{P}_n$ was proved. We now calculate all such polynomials explicitly.

3.1. The SO(2)-invariant subspaces of $\mathcal{P}_n$

By Lemma 1.11, the polynomial $z \mapsto \Re(\xi z^{n-2j})P_j(2|z|^2-1)$, $\xi \in \mathbb{C}$ can be written as a linear combination of Zernike polynomials

$$2\Re(\xi z^{n-2j})P_j(2|z|^2-1) = \xi z^{n-2j}P_j(2|z|^2-1) + \overline{\xi}\overline{z}^{n-2j}P_j(2|z|^2-1).$$

Thus $\mathcal{P}_n$ as a real vector space can be written as an orthogonal direct sum of subspaces

$$\mathcal{P}_n = \bigoplus_{0 \leqslant j \leqslant \frac{n}{2}} V_j, \quad V_j := \operatorname{span}\{z \mapsto \Re(\xi z^{n-2j})P_j(2|z|^2-1) : \xi \in \mathbb{C}\}, \tag{3.1}$$

where each V_j is invariant under the action of SO(2). Moreover, the summands V_j are *absolutely irreducible* under the action of any subgroup G of SO(2) of order 3 or more, i.e., V_j considered as a complex vector space has no G-invariant subspaces other than 0 and V_j. The polynomials in V_j can be factored into a harmonic homogeneous polynomial of degree $n-2j$ multiplied by a common factor of degree $2j$, and so

$$\dim(V_j) = \begin{cases} 2, & j \neq \frac{n}{2}, \\ 1, & j = \frac{n}{2}. \end{cases}$$

Given the decomposition (3.1) of $\mathcal{P}_n$ into absolutely irreducibles, we have the following example of [4, Theorem 6.18].

Theorem 3.2. *Let G be a finite subgroup of* SO(2) *for which* $\operatorname{span}\{gp : g \in G\} = \mathcal{P}_n$, *for some $p \in \mathcal{P}_n$, i.e., $G = \langle R_{\frac{2\pi}{k}} \rangle$, where either $k \geqslant n+1$ and k is odd, or $k \geqslant 2(n+1)$. If $v = \sum_j v_j$, $v_j \in V_j$, then $\{gv\}_{g\in G}$ is an isometric tight frame for $\mathcal{P}_n$ if and only if*

$$\frac{\|v_j\|^2}{\|v_k\|^2} = \frac{\dim(V_j)}{\dim(V_k)}, \quad 0 \leqslant j, k \leqslant \frac{n}{2}.$$

In particular, $\{R^j_{\frac{\pi}{n+1}} v : 0 \leqslant j \leqslant n\}$ is an orthonormal basis for $\mathcal{P}_n$ if and only if

$$\|v_j\| = \begin{cases} \sqrt{\frac{2}{n+1}}, & j \neq \frac{n}{2}, \\ \sqrt{\frac{1}{n+1}}, & j = \frac{n}{2}. \end{cases} \tag{3.3}$$

From this we obtain our main result. Let P_j be the univariate orthogonal polynomial of degree j in Lemma 1.11, and h_j be given by (1.13).

Theorem 3.4. *Let $v \in \mathcal{P}_n$ be the polynomial with real coefficients defined by*

$$v(x,y) := \frac{1}{\sqrt{n+1}} \sum_{0 \leqslant j \leqslant \frac{n}{2}} \frac{2}{1+\delta_{j,\frac{n}{2}}} \frac{1}{\sqrt{h_j}} \Re(\xi_j z^{n-2j})P_j(2|z|^2-1), \quad z := x+iy, \tag{3.5}$$

where ξ_j *are complex numbers of unit modulus, with* $\xi_{\frac{n}{2}} \in \{-1, 1\}$. *Then* $\{R^j_{\frac{\pi}{n+1}} v\}_{j=0}^n$ *is an orthonormal basis for* $\mathcal{P}_n$, *and all such polynomials are given by* (3.5). *Moreover*

$$f = \frac{n+1}{k} \sum_{j=0}^{k-1} \langle f, R^j_{\frac{2\pi}{k}} v\rangle R^j_{\frac{2\pi}{k}} v = \frac{n+1}{2\pi} \int_0^{2\pi} \langle f, R_\theta v\rangle R_\theta v \, d\theta, \quad \forall f \in \mathcal{P}_n, \tag{3.6}$$

whenever $k \geqslant n+1$ *and* k *is odd, or* $k \geqslant 2(n+1)$.

Proof. By Theorems 3.2 and 2.5, we need only find all elements $v_j \in V_j$ satisfying (3.3). For $j \neq \frac{n}{2}$ the Zernike polynomials of (1.12) are orthogonal, so that

$$\|\Re(\xi z^{n-2j}) P_j(2|z|^2 - 1)\|^2 = \frac{1}{4}\left(\|\xi z^{n-2j} P_j(2|z|^2 - 1)\|^2 + \|\overline{\xi}\overline{z}^{n-2j} P_j(2|z|^2 - 1)\|^2\right)$$
$$= \frac{h_j}{2}|\xi|^2,$$

and all the possible choices for v_j are given by

$$v_j(x, y) = \sqrt{\frac{2}{n+1}} \frac{\sqrt{2}}{\sqrt{h_j}} \Re(\xi_j z^{n-2j}) P_j(2|z|^2 - 1), \quad j \neq \frac{n}{2}, \quad |\xi_j| = 1.$$

For $j = \frac{n}{2}$ (when n is even), we have $\|P_j(2|z|^2 - 1)\|^2 = h_j$, and so we must choose

$$v_j(x, y) = \pm \sqrt{\frac{1}{n+1}} \frac{1}{\sqrt{h_j}} P_j(2|z|^2 - 1)$$
$$= \sqrt{\frac{1}{n+1}} \frac{1}{\sqrt{h_j}} \Re(\xi_j z^{n-2j}) P_j(2|z|^2 - 1), \quad j = \frac{n}{2}, \quad \xi_{\frac{n}{2}} \in \{-1, 1\}.$$

Thus $v = \sum_j v_j$ is given by (3.5). $\square$

Example 1. The quadratics $\mathcal{P}_2$. Writing

$$\xi_0 = a + ib, \quad a, b \in \mathbb{R}, \quad a^2 + b^2 = 1,$$

and using (1.14) and (1.15), we have

$$v(x, y) = \frac{2}{\sqrt{3}\sqrt{m_2}}(a(x^2 - y^2) - 2bxy) \pm \frac{\sqrt{m_0}}{\sqrt{3}\sqrt{m_0 m_2 - m_1^2}}\left(x^2 + y^2 - \frac{m_1}{m_0}\right).$$

This formula can yield a ridge function if and only if

$$\frac{2}{\sqrt{m_2}} = \frac{\sqrt{m_0}}{\sqrt{m_0 m_2 - m_1^2}} \iff 3m_0 m_2 = 4m_1^2,$$

in which case each such polynomial is a ridge function

$$v(x,y) = \frac{2}{\sqrt{3}\sqrt{m_2}}(a(x^2-y^2)-2bxy) \pm \frac{2}{\sqrt{3}\sqrt{m_2}}\left(x^2+y^2-\frac{m_1}{m_0}\right)$$

$$= \pm\frac{4}{\sqrt{3m_2}}\left\{\left(\frac{\sqrt{1\pm a}}{\sqrt{2}}x \mp \operatorname{sign}(b)\frac{\sqrt{1\mp a}}{\sqrt{2}}y\right)^2 - \frac{m_1}{2m_0}\right\}.$$

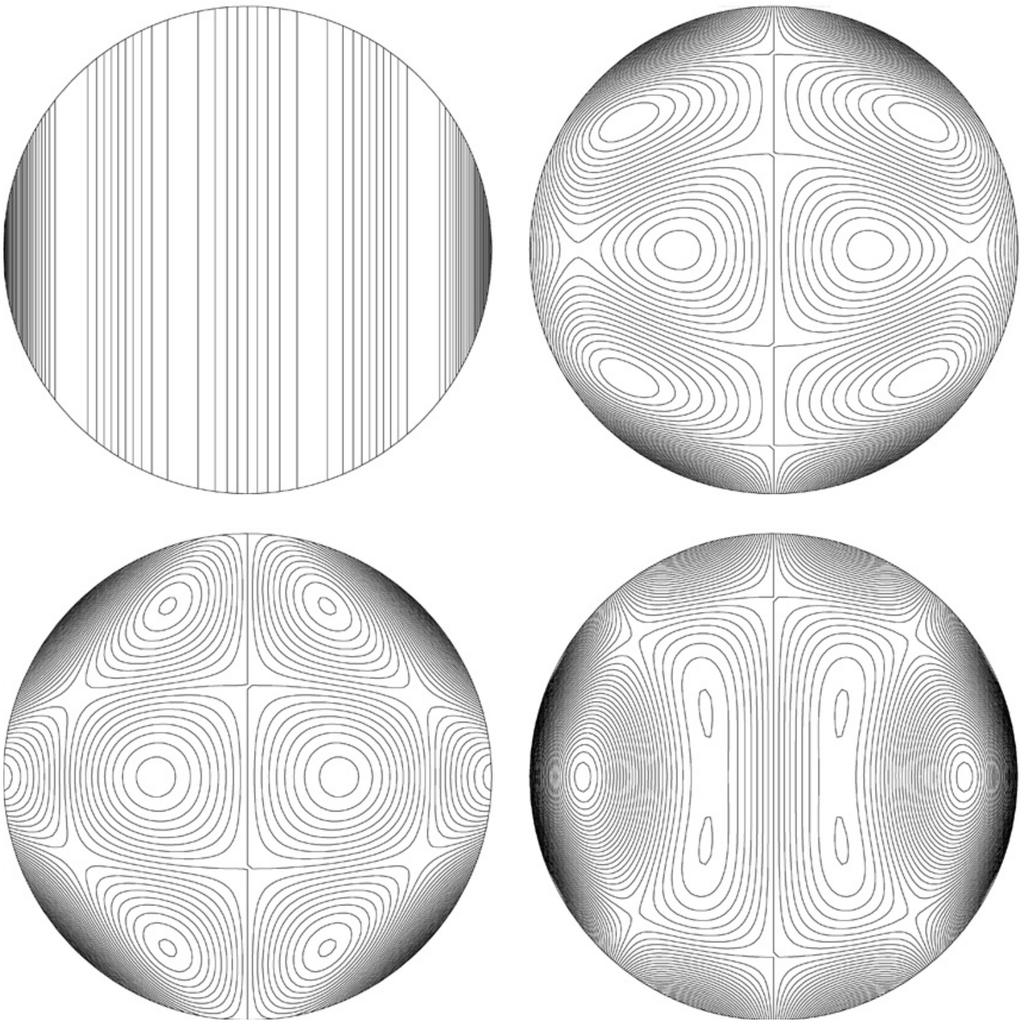

Fig. 1. Contour plots of the quintic Legendre polynomials $v \in \mathcal{P}_5$ given by (3.5) for the choices $\xi_0 = 1$ and $\xi_1, \xi_2 \in \{-1, 1\}$. The first is the Logan–Shepp polynomial.

Example 2. The cubics $\mathcal{P}_3$. Writing

$$\xi_0 = a + ib, \quad \xi_1 = c + id, \quad a, b, c, d \in \mathbb{R}, \quad a^2 + b^2 = c^2 + d^2 = 1,$$

we obtain

$$v(x,y) = \frac{1}{\sqrt{m_3}}(a(x^3-3xy^2)+b(y^3-3x^2y)) + \frac{\sqrt{m_1}}{\sqrt{m_1m_3-m_2^2}}(cx-dy)\left(x^2+y^2-\frac{m_2}{m_1}\right).$$

For the Gegenbauer polynomials, we single out the particular choice for v which corresponds to the decomposition of Logan and Shepp [3] for Legendre polynomials.

Corollary 3.7. *Let $\mathcal{P}_n$ be the Gegenbauer polynomials for the weight given by*

$$w(r) = (1-r^2)^\alpha, \quad \alpha > -1,$$

and $v \in \mathcal{P}_n$ be the Gegenbauer polynomial with real coefficients defined by

$$v(x,y) := \frac{2}{\sqrt{\pi}} \frac{\sqrt{\alpha+n+1}}{\sqrt{n+1}} \sum_{0 \leqslant j \leqslant \frac{n}{2}} \frac{1}{1+\delta_{j,\frac{n}{2}}} \frac{\sqrt{(\alpha+j+1)_{n-2j}}}{\sqrt{(j+1)_{n-2j}}} \times \Re(z^{n-2j}) P_j^{(\alpha,n-2j)}(2|z|^2-1). \tag{3.8}$$

Then $\{R^j_{\frac{\pi}{n+1}} v\}_{j=0}^n$ is an orthonormal basis for $\mathcal{P}_n$. For the Legendre polynomials ($\alpha = 0$), this polynomial reduces to

$$v(x,y) = \frac{1}{\sqrt{\pi}} U_n(x).$$

Proof. For the Gegenbauer polynomials, by (1.16), we have

$$P_j = P_j^{(\alpha,n-2j)}, \quad h_j = \frac{\pi}{(\alpha+n+1)} \frac{(j+1)_{n-2j}}{(\alpha+j+1)_{n-2j}}.$$

Hence taking $\xi_j = 1, \forall j$ in (3.5), we obtain

$$v(x,y) = \frac{2}{\sqrt{\pi}} \frac{\sqrt{\alpha+n+1}}{\sqrt{n+1}} \sum_{0 \leqslant j \leqslant \frac{n}{2}} \frac{1}{1+\delta_{j,\frac{n}{2}}} \frac{\sqrt{(\alpha+j+1)_{n-2j}}}{\sqrt{(j+1)_{n-2j}}} \times \Re(z^{n-2j}) P_j^{(\alpha,n-2j)}(2|z|^2-1).$$

We recall the normalisations

$$P_j^{(\alpha,\beta)}(x) = \frac{1}{2^j} \frac{(j+\alpha+\beta+1)_j}{j!} x^j + \text{lower order terms},$$

$$U_n(x) = 2^{2n} \frac{n!(n+1)!}{(2n+1)!} P_n^{(\frac{1}{2},\frac{1}{2})}(x) = 2^n x^n + \text{lower order terms}.$$

For the Legendre polynomials, the leading term of v is

$$\frac{2}{\sqrt{\pi}} \sum_{0 \leqslant j \leqslant \frac{n}{2}} \frac{1}{1+\delta_{j,\frac{n}{2}}} \binom{n}{j} \frac{z^{n-j}\bar{z}^j + z^j \bar{z}^{n-j}}{2}$$

$$= \frac{1}{\sqrt{\pi}} \sum_{j=0}^n \binom{n}{j} z^j \bar{z}^{n-j} = \frac{1}{\sqrt{\pi}} (z+\bar{z})^n = \frac{2^n}{\sqrt{\pi}} x^n,$$

so that v is the orthogonal projection of $(x,y) \mapsto \frac{2^n}{\sqrt{\pi}} x^n$ onto $\mathcal{P}_n$, i.e.,

$$v(x,y) = \frac{1}{\sqrt{\pi}} U_n(x). \qquad \square$$

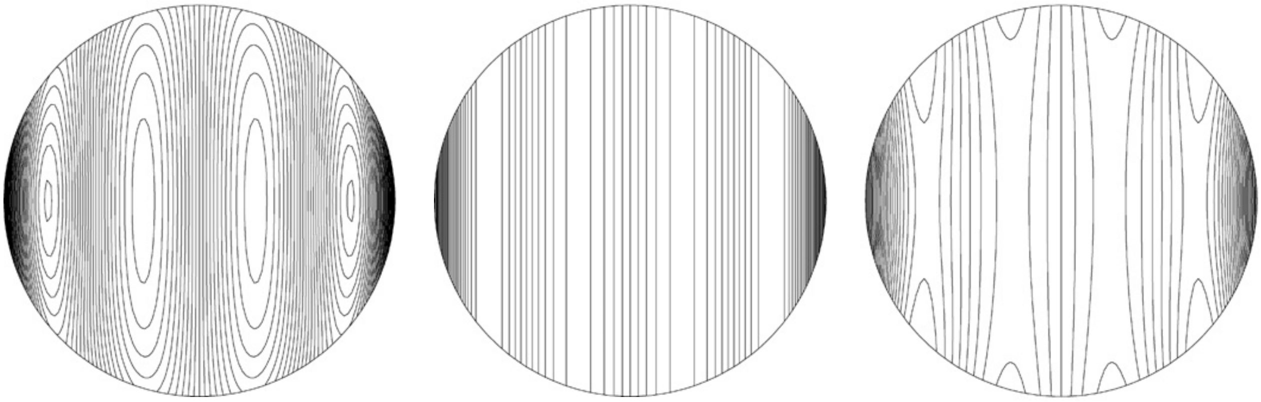

Fig. 2. Contour plots of the $v \in \mathcal{P}_5$ given by (3.8) for $\alpha = -\frac{1}{2}$, 0 (Logan–Shepp), $\frac{1}{2}$.

It is clear that the results presented here have a natural counterpart for orthogonal polynomials on a unit ball in $\mathbb{R}^d$ with a radially symmetric weight, e.g., the integral decomposition in (3.6) would become an integral over the Lie group $\mathrm{SO}(d)$. To give a full generalisation of the results here requires an understanding of those subgroups of $\mathrm{SO}(d)$ which map some homogeneous polynomial of degree n to a basis for this space, and the corresponding numerical integration rule. This is left to the future.

It is hoped that our generalisation of Logan and Shepp's orthogonal expansion will be used to extend important applications based on it to a Gegenbauer weight, e.g., in computed tomography the fast algorithm of [6] for reconstructing images from radon projections.

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