Counting p-groups of exponent p.

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April 2001

Abstract

We count the number of p-groups of exponent p, and order at most p^7, for $p > 5$. Thus the problem reduces to counting nilpotent Lie algebras of dimension at most 7 over $\mathbb{Z}_p$.

1 Dimension 1,2 and 3

We have the abelian Lie algebras of dimension 1,2,3, and $\langle a, b \mid [b, a, a] = [b, a, b] = 0\rangle$, which has class 2 and basis $\{a, b, [b, a]\}$. So there is one of dimension 1, one of dimension 2, and two of dimension 3.

Total is 4.

2 Dimension 4

In addition to the abelian Lie algebra of dimension 4 (4.1), we have two Lie algebras. The ørst is

$$\langle a, b \mid [b, a, a] = 0, \text{ class } 3\rangle \qquad (4.2)$$

which has basis $\{a, b, [b, a], [b, a, b]\}$. The second is

$$\langle a, b \mid \text{class } 2\rangle \oplus \langle c\rangle. \qquad (4.3)$$

This has basis $\{a, b, c, [b, a]\}$, with the generator c lying in the centre.

So the total is 3.

3 Dimension 5

3.1 5 generators

The only 5 generator nilpotent Lie algebra of dimension 5 is the abelian one (5.1).

3.2 4 generators

A four generator nilpotent Lie algebra of dimension 5 must be a descendent of the abelian Lie algebra of dimension 4. There are two possibilities, both of class 2, with the derived algebra of dimension 1. They are

$$\langle a, b \mid \text{class } 2\rangle \oplus \langle c\rangle \oplus \langle d\rangle, \tag{5.2}$$

and

$$\langle a, b \mid \text{class } 2\rangle \oplus_{[a,b]=[c,d]} \langle c, d \mid \text{class } 2\rangle \tag{5.3}$$

3.3 3 generators

A three generator Lie algebra of dimension 5 must be a descendent of the abelian Lie algebra of dimension 3, or a descendent of $\langle a, b \mid \text{class } 2\rangle \oplus \langle c\rangle$.

First we consider descendants of the abelian Lie algebra of dimension 3. Let L be generated by a, b, c. The algebra L must have class 2, and the breadth of L must be 2. So we may assume that L^2 has basis $\{[a, b], [a, c]\}$. So $[b, c] = \lambda[a, b] + \mu[a, c]$ for some λ, μ. But then $[b - \mu a, c + \lambda a] = 0$. Replacing b, c by $b - \mu a, c + \lambda a$, we see that L has a presentation

$$\langle a, b, c \mid [b, c] = 0, \text{ class } 2\rangle. \tag{5.4}$$

Next we consider a descendent of $\langle a, b \mid \text{class } 2\rangle \oplus \langle c\rangle$. Let L be generated by a, b. Then

$$L/L^3 = \langle a, b \mid \text{class } 2\rangle \oplus \langle c\rangle,$$

and L^3 has dimension 1. The subalgebra of L generated by a and b must have class 3 and dimension 4, and we can assume that $[b, a, a] = 0$ and that L^3 is spanned by $[b, a, b]$. We must have $[a, c] = \lambda[b, a, b]$ and $[b, c] = \mu[b, a, b]$ for some λ, μ. Replacing c by $c + \mu[b, a]$ we get $[b, c] = 0$. And then by scaling c we may assume that $\lambda = 0$ or 1. So we have two possibilities here:

$$\langle a, b \mid [b, a, a] = 0, \text{ class } 3\rangle \oplus \langle c\rangle, \tag{5.5}$$

and

$$\langle a, b, c \mid [b, a, a] = [b, c] = 0, [a, c] = [b, a, b], \text{ class } 3\rangle. \tag{5.6}$$

These two Lie algebras are certainly not isomorphic, since in the ørst one c is central, while in the second one the centre is spanned by $[a, b, b]$.

3.4 2 generators

Let L be a 2 generator nilpotent Lie algebra of dimension 5, and let the generators of L be a, b. Then L must be a descendent of $\langle a, b \mid \text{class } 2\rangle$ or of $\langle a, b \mid [b, a, a] = 0,$ class $3\rangle$.

In the ørst case L is the free 2 generator Lie algebra of class 3

$$\langle a, b \mid \text{class } 3\rangle \tag{5.7}$$

In the second case L^3/L^4 must have dimension 1, and we can assume that $[b,a,a] \in L^4$, and that L^4 is spanned by $[b,a,b,b]$. So L has basis

$$\{a, b, [b,a], [b,a,b], [b,a,b,b]\},$$

and L is

$$\langle a, b \mid [b,a,a] = 0, \text{ class } 4\rangle, \tag{5.8}$$

or

$$\langle a, b \mid [b,a,a] = \lambda[b,a,b,b], \text{ class } 4\rangle$$

with $\lambda \neq 0$. Setting $a' = \lambda^{-1}a$ we have

$$[b,a',a'] = \lambda^{-2}[b,a,a] = \lambda^{-1}[b,a,b,b] = [b,a',b,b].$$

So we may assume that $\lambda = 1$ and we have

$$\langle a, b \mid [b,a,a] = [b,a,b,b], \text{ class } 4\rangle. \tag{5.9}$$

It is clear that 5.8 and 5.9 are not isomorphic since in 5.8 the centralizer of L^2 has dimension 4, but in 5.9 it has dimension 3.

4 Dimension 6

4.1 6 generators

The only 6 generator nilpotent Lie algebra of dimension 6 is the abelian one (6.1).

4.2 5 generators

Let L be a 5 generator nilpotent Lie algebra of dimension 6. Then L is a descendent of the abelian Lie algebra of dimension 5, with L^2 of dimension 1. So there are two possibilities for L:

$$\langle a, b \mid \text{class } 2\rangle \oplus \langle c\rangle \oplus \langle d\rangle \oplus \langle e\rangle, \tag{6.2}$$

and

$$\langle a, b \mid \text{class } 2\rangle \oplus_{[a,b]=[c,d]} \langle c, d \mid \text{class } 2\rangle \oplus \langle e\rangle. \tag{6.3}$$

4.3 4 generators

Let L be a four generator nilpotent Lie algebra of dimension 6, and let L be generated by a, b, c, d. Then L must be a descendent of the abelian Lie algebra of dimension 4, or of 5.2. (The algebra 5.3 is terminal.)

First consider the case when L is an immediate descendant of the abelian Lie algebra of dimension 4. Then L^2 has dimension 2 and L must have breadth 2. So we may suppose that a has breadth 2. We can assume that $[a, b]$ and $[a, c]$ form a basis for L^2, and that $[a, d] = 0$. As above we may suppose that $[b, c] = 0$. We have to consider various possibilities for the values of $[b, d]$ and $[c, d]$.

One possibility is that these are both zero, in which case we have

$$\langle a, b, c \mid [b, c] = 0,\ \text{class } 2\rangle \oplus \langle d\rangle. \tag{6.4}$$

Another possibility is that $[b, d]$ and $[c, d]$ span a space of dimension 1. In this case we can assume that $[c, d] = 0$. Then $[b, d] = \lambda[a, b] + \mu[a, c]$ for some λ, μ. Setting $b' = b + \nu c$ we obtain

$$[b', d] = [b, d] = \lambda[a, b] + \mu[a, c] = \lambda[a, b'] + (\mu - \lambda\nu)[a, c].$$

If $\lambda \neq 0$ we can choose ν so that $\mu - \lambda\nu = 0$, and then replacing b by b' and replacing d by $\lambda^{-1}d$ we obtain

$$\langle a, b, c, d \mid [a, d] = [b, c] = [c, d] = 0,\ [b, d] = [a, b],\ \text{class } 2\rangle.$$

We get a slightly simpler presentation for this algebra on generators $a' = a+d$, $b' = c$, $c' = b$, $d' = d$ with basis $[b', a']$, $[d', c']$ for L^2:

$$\langle a, b \mid \text{class } 2\rangle \oplus \langle c, d \mid \text{class } 2\rangle. \tag{6.5}$$

On the other hand if $\lambda = 0$ (and $\mu \neq 0$) then replacing d by $\mu^{-1}d$ we obtain

$$\langle a, b, c, d \mid [a, d] = [b, c] = [c, d] = 0,\ [b, d] = [a, c],\ \text{class } 2\rangle. \tag{6.6}$$

The algebras 6.4, 6.5 and 6.6 are all distinct. First note that d is in the centre of 6.4, but that in 6.5 and 6.6 the centre is the derived algebra. One way to see that 6.5 is not isomorphic to 6.6 is as follows. In 6.5 the elements b, c, d all have breadth one. But in 6.6 we will show that the only elements in the span of a, b, c, d which have breadth one are the elements in the span of c, d.

To see this, consider an element $x = a + \lambda b + \mu c + \nu d$ in 6.6. Then $[x, c] = [a, c]$ and $[x, b] = [a, b] - \nu[a, c]$. So x has breadth 2, and hence any element outside the span of b, c, d has breadth 2. Next consider an element $y = b + \mu c + \nu d$. Then $[y, a] = -[a, b] - \mu[a, c]$ and $[y, d] = [a, c]$. So y has breadth 2 and hence every element outside the span of c, d has breadth 2. The non-zero elements in the span of c, d all have breadth 1.

The ønal possibility is that $[b,d],[c,d]$ span a space of dimension 2. As above, L^2 is spanned by $[a,b],[a,c]$ and $[b,c]=[a,d]=0$. Replacing d by $d+\lambda a$, and then scaling d, we may suppose that $[b,d]=[a,c]$. Let

$$[c,d]=\alpha[a,b]+\beta[a,c],$$

where $\alpha\neq 0$. Let $b'=b+\mu c$. Then

$$\begin{aligned}[b',d] &= \mu\alpha[a,b]+(1+\mu\beta)[a,c],\\ [a,b'] &= [a,b]+\mu[a,c].\end{aligned}$$

So

$$\begin{aligned}[b',d] &= \mu\alpha[a,b']+\gamma[a,c],\\ [c,d] &= \alpha[a,b']+(\beta-\mu\alpha)[a,c].\end{aligned}$$

If we let $\mu=\beta/2\alpha$ and replace d by $d+(\beta/2)a$ then we have

$$\begin{aligned}[b',d] &= \gamma[a,c],\\ [c,d] &= \alpha[a,b'],\end{aligned}$$

where $\alpha,\gamma\neq 0$. Scaling d and replacing b by b' we may assume that $\gamma=1$. So we have $[b,d]=[a,c]$, $[c,d]=\alpha[a,b]$. Now let $a'=\nu a$, $c'=\nu^{-1}c$. Then $[b,d]=[a',c']$ and $[c',d]=\alpha\nu^{-2}[a',b]$. So we can assume that $\alpha=1$ or ω (where ω generates the multiplicative group of $\mathbb{Z}_p$). But if $\alpha=1$ then $[b+c,d]=[a,b+c]$ so that $[b+c,d+a]=0$, and we are back to an earlier case. So we are left with one further possibility:

$$\langle a,b,c,d \mid [a,d]=[b,c]=0,\ [b,d]=[a,c],\ [c,d]=\omega[a,b],\ \text{class } 2\rangle \qquad (6.7)$$

We need to show that 6.7 is distinct from 6.4, 6.5, and 6.6. First not that the centre of 6.7 is its derived algebra, so 6.7 is distinct from 6.4. We show that 6.7 is distinct from 6.5 and 6.6 by counting elements of breadth 1. Consider an element $x=a+\lambda b+\mu c+\nu d$ in 6.7. Then $[x,b]=[a,b]-\nu[a,c]$, $[x,c]=[a,c]-\nu\omega[a,b]$. Since $1-\nu^2\omega\neq 0$, x has breadth 2, and hence any element outside the span of b,c,d has breadth 2. Next consider an element $y=b+\mu c+\nu d$. Then $[y,a]=-[a,b]-\mu[a,c]$ and $[y,d]=[a,c]+\mu\omega[a,b]$. So y has breadth 2 and hence every element outside the span of c,d has breadth 2. Next consider $z=c+\nu d$. Then $[z,a]=-[a,c]$, $[z,d]=\omega[a,b]$, so any element outside the span of d has breadth 2. Finally d has breadth 2, so every element outside the derived algebra has breadth 2.

Now suppose that L is a descendent of 5.2. So

$$L/L^3=\langle a,b \mid \text{class } 2\rangle\oplus\langle c\rangle\oplus\langle d\rangle.$$

Then L^3 must both have dimension 1, and the subalgebra generated by a,b must have class 3 and dimension 4. So we may assume $\langle a,b,c\rangle$ is 5.5 or 5.6, and that the commutators $[a,d],[b,d],[c,d]$ are all scalar multiples of $[b,a,b]$. If these three commutators are all zero then d is central, and we have

$$\langle a,b \,|\, [b,a,a]=0,\ \text{class } 3\rangle \oplus \langle c\rangle \oplus \langle d\rangle, \tag{6.8}$$

or

$$\langle a,b,c \,|\, [b,a,a]=[b,c]=0,\ [a,c]=[b,a,b],\ \text{class } 3\rangle \oplus \langle d\rangle. \tag{6.9}$$

Let us consider the case when d is not central and

$$\langle a,b,c\rangle = \langle a,b \,|\, [b,a,a]=0,\ \text{class } 3\rangle \oplus \langle c\rangle.$$

If $[c,d]=0$ then c is central so that interchanging c and d we have 6.9. If $[c,d]\neq 0$ then we can assume that $[c,d]=[b,a,b]$. We can replace a by $a+\lambda c$ and replace b by $b+\mu c$ so that $[a,d]=[b,d]=0$. So we have a central product

$$\langle a,b \,|\, [b,a,a]=0,\ \text{class } 3\rangle \oplus_{[b,a,b]=[c,d]} \langle c,d \,|\, \text{class } 2\rangle. \tag{6.10}$$

Finally, consider the case when d is not central and when

$$\langle a,b,c\rangle = \langle a,b,c \,|\, [b,a,a]=[b,c]=0,\ [a,c]=[b,a,b],\ \text{class } 3\rangle.$$

Replacing d by $d+\lambda c+\mu[b,a]$ we may assume that $[a,d]=[b,d]=0$. Since we are assuming that d is not central, we may suppose that $[c,d]=[b,a,b]$. So we appear to have one more Lie algebra

$$\begin{aligned}\langle a,b,c,d \,|\, [b,a,a] &= [b,c]=0,\ [a,c]=[b,a,b],\\ [a,d] &= [b,d]=0,\ [c,d]=[b,a,b],\ \text{class } 3\rangle.\end{aligned}$$

However in this algebra, $a+d$ and b are both centralized by c and d and the map $a\mapsto a+d,\ b\mapsto b,\ c\mapsto c,\ d\mapsto d$ gives an isomorphism from 6.10 to this algebra.

4.4 3 generator

Let L be a three generator nilpotent Lie algebra of dimension 6, and let L be generated by a,b,c. Then L is a descendent of the abelian Lie algebra of dimension 3, or of $\langle a,b \,|\, \text{class } 2\rangle \oplus \langle c\rangle$, or of 5.4, 5.5, 5.6.

4.4.1 case 1

If L is a descendent of the abelian Lie algebra of dimension 3 then L is

$$\langle a,b,c \,|\, \text{class } 2\rangle. \tag{6.11}$$

4.4.2 case 2

Now let L be a descendent of $\langle a,b \mid \text{class } 2\rangle \oplus \langle c\rangle$. Then the subalgebra generated by a,b must be the free class 3 Lie algebra on two generators and $[a,c]$ and $[b,c]$ are linear combinations of $[b,a,a]$ and $[b,a,b]$. If $[a,c]=[b,c]=0$ then we have

$$\langle a,b \mid \text{class } 3\rangle \oplus \langle c\rangle. \tag{6.12}$$

If $[a,c]$ and $[b,c]$ are linearly dependent then we can assume that $[b,c]=0$. Suppose that

$$[a,c]=\lambda[b,a,a]+\mu[b,a,b].$$

If $\lambda=0$ then replacing c by $\mu^{-1}c$ we have $[a,c]=[b,a,b]$. If $\lambda\neq 0$ then replacing a by $a'=\lambda a+\mu b$ then

$$[a',c]=\lambda[a,c]=\lambda[b,a,a']=[b,a',a'].$$

So we have

$$\langle a,b,c \mid [b,c]=0,\ [a,c]=[b,a,b],\ \text{class } 3\rangle \tag{6.13}$$

and

$$\langle a,b,c \mid [b,c]=0,\ [a,c]=[b,a,a],\ \text{class } 3\rangle. \tag{6.14}$$

In 6.12 the centre is spanned by $c,[b,a,a],[b,a,b]$ but in 6.13 and 6.14 the centre is spanned by $[b,a,a],[b,a,b]$. So 6.12 is not isomorphic to 6.13 or 6.14. Consider an element

$$\alpha_1 a+\alpha_2 b+\alpha_3 c+\alpha_4[b,a]+\alpha_5[b,a,a]+\alpha_6[b,a,b]\in L.$$

If $\alpha_1=0$ then this element has breadth at most 2 in both 6.13 and 6.14. And if α_1 and α_2 are both non-zero then it has breadth 3 in both algebras. Suppose that $\alpha_1\neq 0$ and $\alpha_2=0$. Then the element has breadth 3 in 6.13 but breadth 2 in 6.14. So 6.13 has more elements of breadth 3, and the two algebras are not isomorphic.

The next case to consider is when

$$L/L^3=\langle a,b \mid \text{class } 2\rangle \oplus \langle c\rangle,$$

with L^3 spanned by $[b,a,a],[b,a,b]$ and with $[a,c],[b,c]$ linearly independent. Replacing c by $c+\lambda[b,a]$ we may assume that $[a,c]=\mu[b,a,b]$. Since $[a,c],[b,c]$ are linearly independent, $\mu\neq 0$ and replacing c by $\mu^{-1}c$ we have $[a,c]=[b,a,b]$. Let $[b,c]=\alpha[b,a,a]+\beta[b,a,b]$. Note that $\alpha\neq 0$ since $[a,c],[b,c]$ are linearly independent. Let $a'=\gamma a,\ b'=\delta b,\ c'=\varepsilon c$. Then

$$[a',c']=\gamma\varepsilon[a,c]=\gamma\varepsilon[b,a,b]=\frac{\varepsilon}{\delta^2}[b',a',b']$$

and

$$[b',c']=\delta\varepsilon[b,c]=\alpha\delta\varepsilon[b,a,a]+\beta\delta\varepsilon[b,a,b]=\frac{\alpha\varepsilon}{\gamma^2}[b',a',a']+\frac{\beta\varepsilon}{\gamma\delta}[b',a',b'].$$

Taking $\varepsilon = \delta^2$ we have $[a', c'] = [b', a', b']$ and

$$[b', c'] = \frac{\alpha\delta^2}{\gamma^2}[b', a', a'] + \frac{\beta\delta}{\gamma}[b', a', b'].$$

If $\beta = 0$ then we can assume that $\alpha\delta^2/\gamma^2 = 1$ or ω, where ω is a generator for the multiplicative group of non-zero elements in $\mathbb{Z}_p$. But if $\alpha\delta^2/\gamma^2 = 1$ then

$$[a' + b', c'] = [b', a', a'] + [b', a', b'] = [b', a', a' + b']$$

and replacing c' by $c' + [b', a']$ we have $[a' + b', c'] = 0$. This contradicts the assumption that $[a, c], [b, c]$ are linearly independent. So one possibility is

$$\langle a, b, c \mid [a, c] = [b, a, b],\ [b, c] = \omega[b, a, a],\ \text{class } 3\rangle. \tag{6.15}$$

We show that this is the only possibility as follows. Let $[a, c] = [b, a, b]$ and $[b, c] = \alpha[b, a, a] + \beta[b, a, b]$ where $\alpha, \beta \neq 0$. Let $a' = a + \lambda b$. Then

$$\begin{aligned}
[b, a', a'] &= [b, a, a] + \lambda[b, a, b], \\
[b, a', b] &= [b, a, b], \\
[a', c] &= [a, c] + \lambda[b, c] = \lambda\alpha[b, a, a] + (1 + \lambda\beta)[b, a, b].
\end{aligned}$$

So

$$\begin{aligned}
[a', c] &= \lambda\alpha[b, a', a'] + (1 + \lambda\beta - \lambda^2\alpha)[b, a', b], \\
[b, c] &= \alpha[b, a', a'] + (\beta - \alpha\lambda)[b, a', b].
\end{aligned}$$

If we let $\lambda = \beta/2\alpha$ then we have

$$\begin{aligned}
[a', c] &= \frac{\beta}{2}[b, a', a'] + \gamma[b, a', b], \\
[b, c] &= \alpha[b, a', a'] + \frac{\beta}{2}[b, a', b]
\end{aligned}$$

for some γ. Now let $c' = c + (\beta/2)[b, a']$. Then

$$\begin{aligned}
[a', c'] &= \gamma[b, a', b], \\
[b, c] &= \alpha[b, a', a'].,
\end{aligned}$$

and by suitable scaling we may take $\gamma = 1$ and $\alpha = \omega$, as above.

4.4.3 case 3

Let L be a descendent of 5.4. So $L/L^3 = \langle a, b, c \mid [b, c] = 0, \text{class } 2\rangle$ and L^3 is spanned by one of

$$[a, b, a], [a, c, a], [a, b, b],\ [a, c, c],\ [a, b, c] = [a, c, b].$$

First, suppose that one of $[a,b,b],[a,b,c],[a,c,c]$ is non-zero. Since $[a,b{+}c,b{+}c]=2[a,b,c]$ we may (if necessary) replace b by c or $b+c$ so that L^3 is spanned by $[a,b,b]$. Then we can choose c to centralize $[a,b]$ so that $[a,b,c]=[a,c,b]=0$. We have $[b,c]=\alpha[a,b,b]$ for some α. Replacing c by $c+\alpha[a,b]$ we may suppose that $[b,c]=0$.

Let $a'=a+\lambda b+\mu c$. Then $[a',b,a']=[a,b,a]+\lambda[a,b,b]$ and $[a',c,a']=[a,c,a]+\mu[a,c,c]$. So we can choose a so that $[a,b,a]=0$ and if $[a,c,c]\neq 0$ we can also choose a so that $[a,c,a]=0$.

If $[a,c,c]=0$ then we can scale c so that $[a,c,a]=0$ or $[a,b,b]$. So we have

$$\langle a,b,c\,|\,[b,c]=[a,b,a]=[a,c,a]=[a,b,c]=[a,c,c]=0,\ \text{class } 3\rangle, \tag{6.16}$$

or

$$\langle a,b,c\,|\,[b,c]=[a,b,a]=[a,b,c]=[a,c,c]=0,\ [a,c,a]=[a,b,b],\ \text{class } 3\rangle. \tag{6.17}$$

If $[a,c,c]\neq 0$ then $[a,c,c]=\beta[a,b,b]$ for some $\beta\neq 0$. Scaling c we may suppose that $\beta=1$, or ω (where ω generates the multiplicative group of $\mathbb{Z}_p$). So we have

$$\langle a,b,c\,|\,[b,c]=[a,b,a]=[a,c,a]=[a,b,c]=0,\ [a,c,c]=[a,b,b],\ \text{class } 3\rangle, \tag{6.18}$$

or

$$\langle a,b,c\,|\,[b,c]=[a,b,a]=[a,c,a]=[a,b,c]=0,\ [a,c,c]=\omega[a,b,b],\ \text{class } 3\rangle. \tag{6.19}$$

Next suppose that $[a,b,b]=[a,c,c]=[a,b,c]=0$. Then one of $[a,b,a],[a,c,a]$ must be non-zero and we may suppose that L^3 is spanned by $[a,b,a]$ and that $[a,c,a]=0$. We have $[b,c]=\gamma[a,b,a]$ and scaling c we may assume that $\gamma=0$ or 1. So we have

$$\langle a,b,c\,|\,[b,c]=[a,b,b]=[a,c,c]=[a,b,c]=[a,c,a]=0,\ \text{class } 3\rangle, \tag{6.20}$$

or

$$\langle a,b,c\,|\,[a,b,b]=[a,c,c]=[a,b,c]=[a,c,a]=0,\ [b,c]=[a,b,a],\ \text{class } 3\rangle. \tag{6.21}$$

We need to show that the six algebras 6.16 - 6.21 are all distinct. In 6.16 the centralizer of L^2 is spanned by a,c modulo L^2, in 6.17 it is spanned by c, in 6.18 and 6.19 it is spanned by a, and in 6.20 and 6.21 it is spanned by b,c modulo L^2. Let C be the centralizer of L^2. Then C has dimension 4 in 6.17, 6.18, 6.19, but dimension 5 in the other three algebras. Also, $[C,L]$ has dimension 2 in 6.17, but dimension 3 in the other øve algebras. So 6.17 is distinct from the other øve algebras. In 6.16 $[C,C]$ is not contained in L^3, but in 6.20 and 6.21 it is. So 6.16 is distinct from the other øve algebras. In 6.20 C is abelian, but in 6.21 it is not. So 6.20 and 6.21 are distinct from each other and from the other four algebras. It remains to prove that 6.18 is not isomorphic to 6.19. Consider a possible automorphism of 6.18. It must map C to C, and so it must map a to $a'=\lambda a+d$ for some $\lambda\neq 0$ and some $d\in L^2$. Let b',c'

be the images of b and c under the automorphism. Then b', c' must generate a two dimensional abelian subalgebra with trivial intersection with L^2. So

$$\begin{aligned} b' &= \beta b + \gamma c + e, \\ c' &= \delta b + \varepsilon c + f, \end{aligned}$$

for some scalars $\beta, \gamma, \delta, \varepsilon$ with $\beta\varepsilon \neq \gamma\delta$ and some $e, f \in L^2$. Also e and f must be chosen so that $[b', c'] = 0$. It is straightforward to see that $[a', b', a'] = [a', c', a'] = 0$. We also require that $[a', b', c'] = 0$. Now

$$[a', b', c'] = \lambda(\beta\delta + \gamma\varepsilon)[a, b, b],$$

and so we require $\beta\delta + \gamma\varepsilon = 0$. Also

$$\begin{aligned} [a', b', b'] &= \lambda(\beta^2 + \gamma^2)[a, b, b], \\ [a', c', c'] &= \lambda(\delta^2 + \varepsilon^2)[a, b, b]. \end{aligned}$$

We want to show that it is impossible to choose a', b', c' in such a way that the above conditions are satisøed and

$$[a', c', c'] = \omega[a', b', b'],$$

and this would require $\delta^2 + \varepsilon^2 = \omega(\beta^2 + \gamma^2) \neq 0$. The conditions $\beta\varepsilon \neq \gamma\delta$, $\beta\delta + \gamma\varepsilon = 0$ imply that if $\beta = 0$ if and only if $\varepsilon = 0$ and that if this holds then $\gamma, \delta \neq 0$. But then we require $\delta^2 = \omega\gamma^2$ which is impossible. So we may suppose that $\beta \neq 0$. Then $\delta = -\gamma\varepsilon\beta^{-1}$ and $\delta^2 + \varepsilon^2 = \varepsilon^2(\beta^2 + \gamma^2)\beta^{-2}$. So we require

$$\varepsilon^2(\beta^2 + \gamma^2)\beta^{-2} = \omega(\beta^2 + \gamma^2) \neq 0.$$

But this would imply that $\omega = \varepsilon^2\beta^{-2}$, which is impossible.

4.4.4 case 4

Let L be a descendent of 5.5. So $L/L^4 = \langle a, b \,|\, [b, a, a] = 0, \text{ class } 3\rangle \oplus \langle c\rangle$. Then L^4 has dimension1 and is spanned by $[b, a, b, b]$. The subalgebra generated by a, b is a øve dimensional descendent of $\langle a, b \,|\, [b, a, a] = 0, \text{ class } 3\rangle$, so we may suppose that $[b, a, a] = 0$ or $[b, a, b, b]$. We have $[a, c] = \lambda[b, a, b, b]$ and $[b, c] = \mu[b, a, b, b]$. If $\lambda \neq 0$ then we can scale c so that $\lambda = 1$, and replace b by $b + \nu a$ so that $[b, c] = 0$. So we have

$$\langle a, b, c \,|\, [b, a, a] = 0,\ [a, c] = [b, a, b, b],\ [b, c] = 0, \text{ class } 4\rangle, \qquad (6.22)$$

or

$$\langle a, b, c \,|\, [b, a, a] = [b, a, b, b],\ [a, c] = [b, a, b, b],\ [b, c] = 0, \text{ class } 4\rangle. \qquad (6.23)$$

On the other hand, if $[a,c]=0$ then we can scale c so that $[b,c]=0$ or $[b,a,b,b]$. But if $[b,c]=[b,a,b,b]$ then replacing c by $c+[b,a,b]$ gives $[b,c]=0$, while preserving $[a,c]=0$. So this gives

$$\langle a,b \mid [b,a,a]=0,\ \text{class } 4\rangle \oplus \langle c\rangle, \tag{6.24}$$

$$\langle a,b \mid [b,a,a]=[b,a,b,b],\ \text{class } 4\rangle \oplus \langle c\rangle. \tag{6.25}$$

Clearly 6.22 and 6.23 have centres of dimension 1, and 6.24 and 6.25 have centres of dimension 2. In 6.22 and 6.24 the centralizer of L^2 has dimension 5, but in 6.23 and 6.25 the centralizer of L^2 has dimension 4. So these four algebras are distinct.

4.4.5 case 5

Let L be a descendent of 5.6. So

$$L/L^4=\langle a,b,c \mid [b,a,a]=[b,c]=0,\ [a,c]=[b,a,b],\ \text{class } 3\rangle.$$

Then L^4 has dimension 1 and is spanned by $[b,a,b,b]$. The subalgebra generated by a,b is a 5 dimensional algebra of class 4, and so we may suppose that $[b,a,a]=0$ or $[b,a,b,b]$. We have $[b,c]=\lambda[b,a,b,b]$ and $[a,c]=[b,a,b]+\mu[b,a,b,b]$ for some λ,μ. Replacing c by $c+\lambda[b,a,b]$ we may suppose that $[b,c]=0$. If $\mu=0$ we have

$$\langle a,b,c \mid [b,a,a]=[b,c]=0,\ [a,c]=[b,a,b],\ \text{class } 4\rangle, \tag{6.26}$$

or

$$\langle a,b,c \mid [b,a,a]=[b,a,b,b],\ [b,c]=0,\ [a,c]=[b,a,b],\ \text{class } 4\rangle.$$

However, if we replace a by $a+c$ in this last algebra, we obtain 6.26.

If $\mu\neq 0$, let $b'=b-\nu c$. Then

$$\begin{aligned}
[b',c] &= 0\\
[b',a] &= [b,a]+\nu[b,a,b]+\mu\nu[b,a,b,b]\\
[b',a,a] &= [b,a,a]\\
[b',a,b'] &= [b,a,b]+\nu[b,a,b,b]-\nu[b,a,c]\\
&= [b,a,b]+2\nu[b,a,b],\\
[b',a,b',b'] &= [b,a,b,b].
\end{aligned}$$

So if we let $\nu=\mu/2$ and we have $[b',c]=0$, $[a,c]=[b',a',b']$, $[b',a,a]=0$ or $[b',a,b',b']$. So the case $\mu\neq 0$ reduces to the case $\mu=0$ dealt with above.

4.5 2 generators

Let $L=\langle a,b\rangle$ be a two generator nilpotent Lie algebra of dimension 6. Then L must have class at least 4, and L/L^4 must have dimension 4 or 5. So L is a descendent of $\langle a,b \mid [b,a,a]=0, \text{class } 3\rangle$ or one of 5.7, 5.8, 5.9. However $\langle a,b \mid [b,a,a]=0, \text{class } 3\rangle$ does not have an immediate descendent of dimension 6, so L is a descendent of one of 5.7, 5.8, 5.9.

4.5.1 case 1

Suppose that L is a descendent of 5.7. Then $L/L^4 = \langle a, b \,|\, \text{class } 3\rangle$, and L^4 has dimension 1.

First consider the case when some element of $L^3 \backslash L^4$ is central. We may suppose that $[b, a, a]$ is central, and that L^4 is spanned by $[b, a, b, b]$. So we have

$$\langle a, b \,|\, [b, a, a, a] = [b, a, a, b] = 0, \text{ class } 4\rangle. \tag{6.27}$$

Now consider the case when no element of $L^3 \backslash L^4$ is central. Since

$$[b, a + b, a + b, a + b] = [b, a, a, a] + 2[b, a, a, b] + [b, a, b, b],$$

interchanging a and b, or replacing a by $a + b$, if necessary, we may suppose that $[b, a, a, a] \neq 0$. Then we can choose b so that $[b, a, a, b] = [b, a, b, a] = 0$. Since $[b, a, b]$ is not central, we have $[b, a, b, b] = \lambda[b, a, a, a]$ for some $\lambda \neq 0$. By scaling a we may assume that $\lambda = 1$ or ω (where ω generates the multiplicative group of $\mathbb{Z}_p$). so we have

$$\langle a, b \,|\, [b, a, a, b] = 0, \ [b, a, b, b] = [b, a, a, a], \text{ class } 4\rangle, \tag{6.28}$$

or

$$\langle a, b \,|\, [b, a, a, b] = 0, \ [b, a, b, b] = \omega[b, a, a, a], \text{ class } 4\rangle. \tag{6.29}$$

Clearly 6.27 is distinct from 6.28 and 6.29 since the centre of 6.27 has dimension 2 while the centre of 6.28 and 6.29 has dimension 1. The proof that 6.28 and 6.29 are not isomorphic is essentially the same as the proof that 6.18 and 6.19 are not isomorphic.

4.5.2 case 2

Next suppose that L is a descendent of 5.8. So

$$L/L^5 = \langle a, b \,|\, [b, a, a] = 0, \text{ class } 4\rangle$$

and L^5 has dimension 1, and is spanned by $[b, a, b, b, b]$ or $[b, a, b, b, a]$.

First suppose that $[b, a, b, b, a] = 0$. Then L^5 has basis $[b, a, b, b, b]$ and $[b, a, a] = \lambda[b, a, b, b, b]$ for some λ. Scaling a we may suppose that $\lambda = 0$ or 1. So we have

$$\langle a, b \,|\, [b, a, a] = [b, a, b, b, a] = 0, \text{ class } 5\rangle, \tag{6.30}$$

or

$$\langle a, b \,|\, [b, a, a] = [b, a, b, b, b], \ [b, a, b, b, a] = 0, \text{ class } 5\rangle, \tag{6.31}$$

Next suppose that $[b, a, b, b, a] \neq 0$. Then $b + \mu a$ must centralize $[b, a, b, b]$ for some μ. If we set $b' = b + \mu a$ then $[b', a, a] = [b, a, a]$. So, replacing b by b' we may suppose that b centralizes L^4, and that

$$[b, a, a] = \lambda[b, a, b, b, a]$$

for some λ. If we set $a' = a + \lambda[b, a, b]$ then

$$\begin{aligned}
[b, a'] &= [b, a] - \lambda[b, a, b, b] \\
[b, a', b] &= [b, a, b], \\
[b, a', b, b] &= [b, a, b, b], \\
[b, a', b, b, b] &= 0, \\
[b, a', b, b, a] &= [b, a, b, b, a], \\
[b, a', a'] &= [b, a, a] + \lambda[b, a, [b, a, b]] = 0.
\end{aligned}$$

So we may assume that $\lambda = 0$ and that we have

$$\langle a, b \mid [b, a, a] = [b, a, b, b, b] = 0, \text{ class } 5\rangle. \tag{6.32}$$

In 6.30 the centralizer of L^2 is $\langle a\rangle + L^2$, in 6.31 it is L^2, and in 6.32 it is L^4. So these three algebras are distinct.

4.5.3 case 3

Finally suppose that L is a descendent of 5.9. So

$$L/L^5 = \langle a, b \mid [b, a, a] = [b, a, b, b], \text{ class } 4\rangle$$

and L^5 has dimension 1 and is spanned by $[b, a, b, b, a]$ or $[b, a, b, b, b]$.

First suppose that $[b, a, b, b, a] = 0$. Then L^5 has basis $[b, a, b, b, b]$ and $[b, a, a] = [b, a, b, b] + \lambda[b, a, b, b, b]$ for some λ. If we let $b' = b + \mu a$ then

$$\begin{aligned}
[b', a] &= [b, a], \\
[b', a, a] &= [b, a, a], \\
[b', a, b'] &= [b, a, b] + \mu[b, a, a], \\
[b', a, b', b'] &= [b, a, b, b] + \mu[b, a, b, a] + \mu[b, a, a, b] + \mu^2[b, a, a, a] \\
&= [b, a, b, b] + 2\mu[b, a, b, b, b].
\end{aligned}$$

If we let $\mu = \lambda/2$ then $[b', a, a] = [b', a, b', b']$. So L is

$$\langle a, b \mid [b, a, a] = [b, a, b, b], [b, a, b, b, a] = 0, \text{ class } 5\rangle. \tag{6.33}$$

Next suppose that $[b, a, b, b, a] \neq 0$. Then $b + \mu a$ must centralize $[b, a, b, b]$ for some μ. If we set $b' = b + \mu a$ then (as above)

$$\begin{aligned}
[b', a] &= [b, a], \\
[b', a, a] &= [b, a, a], \\
[b', a, b'] &= [b, a, b] + \mu[b, a, a], \\
[b', a, b', b'] &= [b, a, b, b] \bmod L^5.
\end{aligned}$$

So $[b', a, a] = [b', a, b', b']$ mod L^5. So, replacing b by b' we may suppose that b centralizes L^4, and that

$$[b, a, a] = [b, a, b, b] + \lambda[b, a, b, b, a]$$

for some λ. Now let $a' = a + (\lambda/2)[b, a, b]$. Then

$$\begin{aligned}
[b, a'] &= [b, a] - (\lambda/2)[b, a, b, b], \\
[b, a', a'] &= [b, a, a] + (\lambda/2)[b, a, [b, a, b]] - (\lambda/2)[b, a, b, b, a] \\
&= [b, a, b, b], \\
[b, a', b] &= [b, a, b], \\
[b, a', b, b] &= [b, a, b, b], \\
[b, a', b, b, a'] &= [b, a, b, b, a], \\
[b, a', b, b, b] &= 0.
\end{aligned}$$

So replacing a by a' we may suppose that $[b, a, a] = [b, a, b, b]$, $[b, a, b, b, b] = 0$. This gives

$$\langle a, b \mid [b, a, a] = [b, a, b, b],\ [b, a, b, b, b] = 0,\ \text{class } 5\rangle. \tag{6.34}$$

The algebras 6.33 and 6.34 are not isomorphic since the centralizer of L^2 is L^2 in 6.33, but is L^4 in 6.34.

4.6 Reconciliation

Here is a reconciliation between the numbering for dimension 6 algebras given above and the numbering given in David Wilkinson's paper ıThe groups of exponent p and order p^7 (p any prime)ȷ.

1	2	3	4	5	6	7	8	9	10	11	12	13	14
6.1	6.2	6.8	6.4	6.12	6.3	6.9	6.24	6.25	6.11	6.5	6.6	6.7	6.13

15	16	17	18	19	20	21	22	23	24
6.16	6.14	6.20	6.21	6.15	6.27	6.10	6.22	6.23	6.18

25	26	27	28	29	30	31	32	33	34
6.19	6.17	6.29	6.28	6.26	6.30	6.31	6.32	6.33	6.34

Note that in 6.32 the relation $[b, a, a] = 0$ implies that $[b, a, b, a] = 0$ since we are dealing with a Lie algebras. But in group 32 (with a and b interchanged) the relation $[b, a, b] = 1$ does not imply that $[b, a, a, b] = 1$.

5 Dimension 7

The total number of nilpotent Lie algebras of dimension 7 over $\mathbb{Z}_p$ is $7p + 174 + 2\gcd(p-1,3)$. This is valid for all odd p, including $p = 3$. However for $p = 3$, in case 1 of the three generator groups there are 22 algebras, and 23 for $p > 3$, and in case 11 of the three generator groups there are $3 + \gcd(p-1,3)$ groups when $p = 3$, but $2 + \gcd(p-1,3)$ groups when $p > 3$.

5.1 7 generators

The only 7 generator nilpotent Lie algebra of dimension 7 is the abelian one (7.1) [DW.1].

5.2 6 generators

Let L be a 6 generator nilpotent Lie algebra of dimension 7. Then L is a descendent of the abelian Lie algebra of dimension 6, with L^2 of dimension 1. So there are three possibilities for L [DW.2,6,110]:

$$\langle a, b \mid \text{class } 2\rangle \oplus \langle c\rangle \oplus \langle d\rangle \oplus \langle e\rangle \oplus \langle f\rangle, \tag{7.2}$$

and

$$\langle a, b \mid \text{class } 2\rangle \oplus_{[a,b]=[c,d]} \langle c, d \mid \text{class } 2\rangle \oplus \langle e\rangle \oplus \langle f\rangle, \tag{7.3}$$

$$\langle a, b \mid \text{class } 2\rangle \oplus_{[a,b]=[c,d]=[e,f]} \langle c, d \mid \text{class } 2\rangle \oplus_{[a,b]=[c,d]=[e,f]} \langle e, f \mid \text{class } 2\rangle. \tag{7.4}$$

5.3 5 generators

Let L be a øve generator nilpotent Lie algebra of dimension 7. Then L is an immediate descendent of 5.1 or 6.2. (The algebra 6.3 is terminal.)

5.3.1 case 1

Let L be an immediate descendent of 5.1, and let the generators of L be a, b, c, d, e. Then L^2 has dimension 2, so L has breadth 2, and we may suppose that a has breadth 2. We may also suppose that L^2 is spanned by $[a, b], [a, c]$ and that $[a, d] = [a, e] = 0$. So $\langle a, b, c, d\rangle$ is a 4 generator subalgebra of dimension 6, which implies that $\langle a, b, c, d\rangle$ is isomorphic to 6.4, 6.5, 6.6, or 6.7. We get 4 algebras if e is central [DW.4,11,12,13].

$$\langle a, b, c \mid [b, c] = 0,\ \text{class } 2\rangle \oplus \langle d\rangle \oplus \langle e\rangle, \tag{7.5}$$

$$\langle a, b, c, d \mid [a, d] = [b, c] = [c, d] = 0,\ [b, d] = [a, b],\ \text{class } 2\rangle \oplus \langle e\rangle, \tag{7.6}$$

$$\langle a, b, c, d \mid [a, d] = [b, c] = [c, d] = 0,\ [b, d] = [a, c],\ \text{class } 2\rangle \oplus \langle e\rangle, \tag{7.7}$$

$$\langle a, b, c, d \mid [a, d] = [b, c] = 0,\ [b, d] = [a, c],\ [c, d] = \omega[a, b],\ \text{class } 2\rangle \oplus \langle e\rangle. \tag{7.8}$$

For the remaining 5 generator algebras L of dimension 7 we assume that L^2 is spanned by $[a,b],[a,c]$, and that $[a,d]=[a,e]=[b,c]=0$. We divide them into two types – those where $[d,e]=0$ and those where $[d,e]\neq 0$.

First we consider algebras where $[d,e]=0$. We show that these reduce to the situation when e is central (which were dealt with above), and to algebras (with a diœerent choice of generators) in which $[d,e]\neq 0$. We subdivide algebras with $[d,e]=0$ into two further cases – those where b has breadth 2, and those where every non-zero element in the span of $\{b,c\}$ has breadth 1.

So suppose that $[d,e]=0$ and that b has breadth 2. Then b must be centralized by $\alpha a+\beta d+\gamma e$ for some α,β,γ with $\beta\neq 0$ or $\gamma\neq 0$. Replacing e by $\alpha a+\beta d+\gamma e$ may suppose that $[b,e]=0$. If $[c,e]=0$ then e is central in L and L is isomorphic to one of 7.5, 7.6, 7.7, 7.8. On the other hand, if $[c,e]\neq 0$ then b is an element of breadth 2, and the centralizer of b contains b,c,e and so is not abelian. So, replacing a by b we are in the case $[d,e]\neq 0$ which we deal with below.

Next suppose that $[d,e]=0$ and that every non-zero element in the span of $\{b,c\}$ has breadth 1. Then b is centralized by $\alpha a+d$, $\beta a+e$ for some α,β. Replacing d by $\alpha a+d$ and replacing e by $\beta a+e$ we have $[b,d]=[b,e]=0$. Similarly c is centralized by $\gamma a+d$, $\delta a+e$ for some γ,δ. But this implies that c is centralized by some non-zero element e' in the span of $\{d,e\}$. Then e' is in the centre of L, and so replacing e by e' we are in the situation dealt with above.

Now consider the case when $[d,e]\neq 0$. Then $[d,e]=[a,\beta b+\gamma c]$ for some β,γ not both zero. Replacing b by $\beta b+\gamma c$, we may suppose that $[d,e]=[a,b]$.

If b has breadth 1 then, as above, we may suppose that $[b,d]=[b,e]=0$. If c is also centralized by d and e then we have

$$\langle a,b,c\,|\,[b,c]=0,\ \text{class }2\rangle\oplus_{[a,b]=[d,e]}\langle d,e\,|\,\text{class }2\rangle. \tag{7.9}$$

[This is DW.43.] If c is not centralized by both d and e and c has breadth 1 then we may suppose that c is centralized by $\alpha a+d$ and e for some $\alpha\neq 0$. Replacing d by $(1/\alpha)d$ and replacing e by αe we see that we may suppose that c is centralized by $a+d$ and e, while preserving the relation $[d,e]=[a,b]$. This gives a Lie algbebra of class 2 and dimension 7 with generators a,b,c,d,e and relations

$$\begin{aligned}[a,d] &= [a,e]=[b,c]=[b,d]=[b,e]=[c,e]=0,\\ [c,d] &= [a,c],\ [d,e]=[a,b].\end{aligned}$$

However if we let $a'=a$, $b'=b-e$, $c'=c$, $d'=a+d$, $e'=e$, then we see that a',b',c',d',e' satisfy the relations of 7.9, so that this is the same algebra again.

On the other hand, if c has breadth 2 then we may suppose that $\alpha a+d$ centralizes c for some α, and that $[c,e]=\lambda[a,b]+\mu[a,c]$ for some λ,μ with $\lambda\neq 0$. Let $c'=c-\lambda d$. Then c' is centralized by $b,c',\alpha a+d$. And

$$[c',e]=\lambda[a,b]+\mu[a,c]-\lambda[d,e]=\mu[a,c]=\mu[a,c'].$$

So c' has breadth 1, and replacing c by c' we obtain 7.9 or 7.10.

Now suppose that b has breadth 2. We may suppose that $[b,e]=0$ and that $[b,d]=\lambda[a,b]+\mu[a,c]$ for some λ,μ with $\mu\neq 0$. If we let $d'=d+\lambda a$ then $[a,d']=0$, $[d',e]=[a,b]$ and $[b,d']=\mu[a,c]$. So replacing d by d' we have $[b,e]=0$, $[b,d]=\mu[a,c]$. Scaling c can take $\mu=1$.

If c has breadth 1 then $[c,d]=\alpha[a,c]$, $[c,e]=\beta[a,c]$ for some α,β. If $\beta\neq 0$ then we let $d'=d-(\alpha/\beta)e$, so that $[a,d']=0$, $[d',e]=[a,b]$, $[b,d']=\mu[a,c]$, $[c,d']=0$. So replacing d by d' we may suppose that $\alpha=0$. In other words, we may suppose that at least one of α,β is zero. If $\alpha=\beta=0$ then we have an algebra [DW.44] with relations

$$\begin{aligned}[a,d] &= [a,e]=[b,c]=[b,e]=[c,d]=[c,e]=0,\\ [b,d] &= [a,c],\ [d,e]=[a,b].\end{aligned} \tag{7.10}$$

If $\alpha=0$ and $\beta\neq 0$ then we have an algebra with relations

$$\begin{aligned}[a,d] &= [a,e]=[b,c]=[b,e]=[c,d]=0,\\ [b,d] &= [a,c],\ [c,e]=\beta[a,c],\ [d,e]=[a,b].\end{aligned}$$

Replacing a by βa and replacing d by βd we obtain

$$\begin{aligned}[a,d] &= [a,e]=[b,c]=[b,e]=[c,d]=0,\\ [b,d] &= [a,c],\ [c,e]=[a,c],\ [d,e]=[a,b].\end{aligned}$$

But if we set $a'=a$, $b'=b+d$, $c'=c$, $d'=d$, $e'=a+e$, then we see that a',b',c',d',e' satisfy the relations of 7.10, so that this is not a new algebra.

If $\alpha\neq 0$ and $\beta=0$ then we have an algebra with relations

$$\begin{aligned}[a,d] &= [a,e]=[b,c]=[b,e]=[c,e]=0,\\ [b,d] &= [a,c],\ [c,d]=\alpha[a,c],\ [d,e]=[a,b].\end{aligned}$$

Now we let $b'=b+\alpha e$, $c'=c-\alpha b$. Then

$$\begin{aligned}[a,d] &= [a,e]=[b',c']=[b',e]=[c',e]=0,\\ [c',d] &= [c,d]-\alpha[b,d]=0,\\ [b',d] &= [b,d]-\alpha[d,e]=[a,c]-\alpha[a,b]=[a,c'],\\ [d,e] &= [a,b],\end{aligned}$$

so that we have algebra 7.10.

Finally suppose that b and c both have breadth 2. As above we may suppose that $[b,d]=[a,c]$, $[b,e]=0$. Let $[c,d]=\lambda[a,b]+\mu[a,c]$, and set $c'=c+\lambda e$. Then $[a,c']=[a,c]$, $[b,c']=0$, $[c',d]=[c,d]-\lambda[d,e]=\mu[a,c]$. So replacing c by c' we may assume that $\lambda=0$. Now let $[c,e]=\alpha[a,b]+\beta[a,c]$. If $\alpha=0$ then c has breadth 1, and that case has already been covered. So we may suppose that $\alpha \neq 0$. If μ and β are both non-zero then we let $c' = c - (\alpha\mu/\beta)e$. We have $[a, c'] = [a, c]$, $[b, c'] = 0$,

$$\begin{aligned} [c', d] &= [c, d] + (\alpha\mu/\beta)[d, e] = (\alpha\mu/\beta)[a, b] + \mu[a, c], \\ [c', e] &= [c, e] = \alpha[a, b] + \beta[a, c]. \end{aligned}$$

So $[c', d]$ and $[c', e]$ are linearly dependent and setting $d' = d - (\mu/\beta)e$ we have $[c', d'] = 0$. We also have

$$[b, d'] = [b, d] = [a, c] = [a, c'],$$

so replacing c by c' and replacing d by d' we may assume that $[c, d] = 0$.

Summarizing, we have $[b, d] = [a, c]$, $[b, e] = 0$, $[c, d] = \mu[a, c]$, $[c, e] = \alpha[a, b] + \beta[a, c]$ where $\alpha \neq 0$ and where at least one of μ, β equals zero.

Consider the case when $\beta = 0$. Let $b' = \alpha a + b + \mu e$, $c' = c - \alpha\mu a - \mu b - \alpha d$. Then $[a, b'] = [a, b]$, $[a, c'] = [a, c] - \mu[a, b]$, and

$$[b', c'] = \alpha[a, c] - \alpha[b, d] - \mu[c, e] + \alpha\mu[a, b] = 0.$$

Furthermore

$$\begin{gathered} [b', d] = [b, d] - \mu[d, e] = [a, c] - \mu[a, b] = [a, c'], \\ [b', e] = [b, e] = 0, \\ [c', d] = [c, d] - \mu[b, d] = 0, \\ [c', e] = [c, e] - \alpha[d, e] = 0, \\ [d, e] = [a, b]. \end{gathered}$$

So we have the algebra 7.10.

Finally consider the case when $[c, d] = 0$, $[c, e] = \alpha[a, b] + \beta[a, c]$, with $\alpha \neq 0$. Let $b' = \alpha a + b + \beta d$, $c' = c - \alpha d$, $e' = \beta a + e$. Then $[a, b'] = [a, b]$, $[a, c'] = [a, c]$, $[a, d] = [a, e'] = 0$, and $[d, e'] = [a, b']$. We also have

$$\begin{gathered} [b', c'] = \alpha[a, c] - \alpha[b, d] = 0, \\ [b', d] = [b, d] = [a, c] = [a, c'], \\ [b', e'] = -\beta[a, b] + \beta[d, e] = 0, \\ [c', d] = 0, \\ [c', e'] = -\beta[a, c] + [c, e] + \alpha[d, e] = 0. \end{gathered}$$

So we have the algebra 7.10 again.

It remains to show that the algebras 7.5 ˘ 7.10 are all distinct. The algebras 7.5 ˘ 7.8 are of the form $L \oplus \langle e \rangle$ where L is one of the algebras 6.4 ˘ 6.7. So these algebras are all distinct. And it is easy to see that the centre of the algebras 7.9 and 7.10 are the derived algebras, so the algebras 7.9 and 7.10 are diœerent from 7.5 ˘ 7.8. To show that 7.9 is not isomorphic to 7.9 we count the number of elements of breadth 2 in the two algebra. Algebra 7.9 has $(p-1)p^3(p^3+p^2-1)$ elements of breadth 2, and 7.10 has $(p-1)p^3(p^3+p^2+p)$ such elements. So these two algebras cannot be isomorphic either.

5.3.2 Case 2

Let L be an immediate descendent of 6.2. Recall that 6.2 is the algebra

$$\langle a, b \mid \text{class } 2\rangle \oplus \langle c\rangle \oplus \langle d\rangle \oplus \langle e\rangle.$$

So L^2 is generated modulo L^3 by $[b,a]$, and $[a,c]$, $[a,d]$, $[a,e]$, $[b,c]$, $[b,d]$, $[b,e]$, $[c,d]$, $[c,e]$, $[d,e]$ all lie in L^3. The Jacobi identity implies that c,d,e all centralize $[b,a]$, and so L^3 is spanned by $[b,a,a]$, $[b,a,b]$. Since L^3 has dimension 1, we can choose a,b so that $[b,a,b]=0$ and L^3 is spanned by $[b,a,a]$. So the centralizer of L^2 is $\langle b,c,d,e\rangle + L^2$, and the centralizer of L modulo L^3 is $\langle c,d,e\rangle + L^2$.

First, we show that we may assume that $[a,c]=[a,d]=[a,e]=0$. Suppose that $[a,c]=\alpha[b,a,a]$, and let $c' = c+\alpha[b,a]$. Then $[a,c']=0$, and we replace c by c'. In other words, we can assume that $[a,c]=0$. Similarly we can assume that $[a,d]=[a,e]=0$.

If $\langle b,c,d,e\rangle + L^2$ is abelian then we have the algebra [DW.3]

$$\langle a, b \mid [b,a,b]=0,\ \text{class } 3\rangle \oplus \langle c\rangle \oplus \langle d\rangle \oplus \langle e\rangle. \tag{7.11}$$

If $\langle b,c,d,e\rangle + L^2$ is not abelian, but $\langle c,d,e\rangle + L^2$ is abelian then we may assume that d,e centralize b and that $[b,c]=\alpha[b,a,a]$ for some $\alpha\neq 0$. By scaling c we can take $\alpha=1$. so we have [DW.7]

$$\langle a,b,c \mid [b,a,b]=[a,c]=0,\ [b,c]=[b,a,a],\ \text{class } 3\rangle \oplus \langle d\rangle \oplus \langle e\rangle. \tag{7.12}$$

If $\langle c,d,e\rangle + L^2$ is not abelian then c,d,e generate a 4 dimensional algebra of class 2, with derived algebra spanned by $[b,a,a]$. So we may assume that $[c,d]=[b,a,a]$, $[c,e]=[d,e]=0$. Replacing b by $b+\lambda c+\mu d$ for suitable λ,μ we can assume that $[b,c]=[b,d]=0$. Then $[b,e]=\nu[b,a,a]$, and by scaling e we can assume that $\nu=0$ or 1. This gives two algebras [DW.21, DW.111]

$$\langle a, b \mid [b,a,b]=0,\ \text{class } 3\rangle \oplus_{[b,a,a]=[c,d]} \langle c, d \mid \text{class } 2\rangle \oplus \langle e\rangle, \tag{7.13}$$

and a class 3 algebra with generators a,b,c,d,e and relations

$$\begin{aligned} [a,c] &= [a,d]=[a,e]=[b,c]=[b,d]=[c,e]=[d,e]=0, \\ [b,a,b] &= 0,\ [b,e]=[c,d]=[b,a,a]. \end{aligned} \tag{7.14}$$

5.4 4 generators

Let L be a four generator nilpotent Lie algebra of dimension 7. Then L is an immediate descendent of the abelian algebra of dimension 4, or of 5.2 or of one of 6.4˘6.10. (The algebra 5.3 is terminal.)

5.4.1 case 1

Let L be an immediate descendant of the abelian algebra of dimension 4. If one of the generators of L is central then L is [DW.10]

$$\langle a, b, c \mid \text{class } 2\rangle \oplus \langle d\rangle. \tag{7.15}$$

So suppose that L is generated by a, b, c, d, but that there are no central elements (other than zero) in the linear span of a, b, c, d. Note that L has class 2 and that L^2 has dimension 3.

First we consider the case when a has breadth 1. We may assume that $[a, b] \neq 0$ and that $[a, c] = [a, d] = 0$. The generator b cannot also have breadth 1, as this would imply that L^2 was spanned by $[a, b]$ and $[c, d]$ which is impossible since L^2 has dimension 3. Suppose that b has breadth 2. Then we may assume that $[a, b]$ and $[b, c]$ are linearly independent, but that $[b, d] = \alpha[a, b] + \beta[b, c]$ for some α, β, and setting $d' = \alpha a - \beta c + d$ we see that L is generated by a, b, c, d' and that $[a, c] = [a, d'] = [b, d'] = 0$. So we have [DW.35]

$$\langle a, b, c, d \mid [a, c] = [a, d] = [b, d] = 0, \text{ class } 2\rangle. \tag{7.16}$$

Next suppose that a has breadth 1 as above (with $[a, c] = [a, d] = 0$), and that b has breadth 3. Then L^2 is spanned by $[a, b]$, $[b, c]$, $[b, d]$ and $[c, d] = \alpha[a, b] + \beta[b, c] + \gamma[b, d]$ for some α, β, γ. If $\gamma \neq 0$ then we set $b' = b - (1/\gamma)c$. Then L is generated by a, b', c, d, $[a, b'] = [a, b]$, $[a, c] = [a, d] = 0$, $[b', c] = [b, c]$, and

$$[b', d] = [b, d] - \frac{1}{\gamma}[c, d] = -\frac{\alpha}{\gamma}[a, b'] - \frac{\beta}{\gamma}[b', c],$$

so that b' has breadth 2. Thus if $\gamma \neq 0$ we have 7.16. Similarly if $\beta \neq 0$ then the element $b' = b + (1/\beta)d$ has breadth 2, and we have 7.16 again. So we may suppose that $[c, d] = \alpha[a, b]$ for some α. By scaling d we have $\alpha = 0$ or 1. This gives [DW.39]

$$\langle a, b, c, d \mid [a, c] = [a, d] = [c, d] = 0, \text{ class } 2\rangle, \tag{7.17}$$

and [DW.37]

$$\langle a, b, c, d \mid [a, c] = [a, d] = 0,\ [c, d] = [a, b], \text{ class } 2\rangle. \tag{7.18}$$

Now we assume that there are no elements of breadth 1 in the linear span of a, b, c, d, but that a has breadth 2. We suppose that $[a, b]$ and $[a, c]$ are linearly independent, but that $[a, d] = 0$. Suppose for the moment that L^2 is spanned by $[a, b]$, $[a, c]$, $[b, c]$.

We write

$$\begin{bmatrix} [b, d] \\ [c, d] \end{bmatrix} = A \begin{bmatrix} [a, b] \\ [a, c] \end{bmatrix} + \begin{bmatrix} \alpha[b, c] \\ \beta[b, c] \end{bmatrix}$$

for some 2×2 matrix A and some α, β.

Let us ørst consider the case when at least one of α, β is non-zero. Then we can pick b', c' in the linear span of b, c so that

$$\begin{bmatrix} [b', d] \\ [c', d] \end{bmatrix} = A \begin{bmatrix} [a, b'] \\ [a, c'] \end{bmatrix} + \begin{bmatrix} 0 \\ [b', c'] \end{bmatrix}$$

for some A. In other words, replacing b, c by b', c' we may assume that $\alpha = 0$ and $\beta = 1$. Let

$$[c, d] = \lambda[a, b] + \mu[a, c] + [b, c].$$

Let $c' = -\lambda a + c$, $d' = \mu a + d$. Then $[a, c'] = ac$, $[a, d'] = 0$, L^2 is spanned by $[a, b]$, $[a, c']$ and $[b, c']$ and

$$[c', d'] = [-\lambda a + c, \mu a + d] = \lambda[a, b] + [b, c] = [b, c'].$$

In addition, $[b, d'] = \gamma[a, b] + \delta[a, c']$ for some γ, δ. Now let $d'' = \gamma a + d'$, $b' = -\gamma a + b$. Then $[a, d''] = 0$, and $[a, b'] = [a, b]$, $[a, c'] = [a, c]$,

$$[b', c'] = [-\gamma a + b, -\lambda a + c] = [b, c] + \lambda[a, b] - \gamma[a, c],$$

$$[b', d''] = [-\gamma a + b, \gamma a + d'] = -\gamma[a, b] + [b, d'] = \delta[a, c'],$$

$$[c', d''] = [c', \gamma a + d'] = -\gamma[a, c'] + [b, c'] = [b', c'].$$

Replacing b, c, d by b', c', d'' we see that L is generated by a, b, c, d with L^2 spanned by $[a, b]$, $[a, c]$ and $[b, c]$. In addition $[a, d] = 0$, $[b, d] = \delta[a, c]$, $[c, d] = [b, c]$. We can assume that $\delta \neq 0$, since otherwise d has breadth 1. Then replacing a by δa we have [DW.36]

$$\langle a, b, c, d \mid [a, d] = 0,\ [b, d] = [a, c],\ [c, d] = [b, c],\ \text{class } 2\rangle. \tag{7.19}$$

So we can assume that $\alpha = \beta = 0$, and that

$$\begin{bmatrix} [b, d] \\ [c, d] \end{bmatrix} = A \begin{bmatrix} [a, b] \\ [a, c] \end{bmatrix}$$

for some 2×2 matrix A. If we let $d' = \lambda a + d$ then $[a, d'] = 0$ and

$$\begin{bmatrix} [b, d'] \\ [c, d'] \end{bmatrix} = (A - \lambda I) \begin{bmatrix} [a, b] \\ [a, c] \end{bmatrix}.$$

We choose λ so that the trace of $A - \lambda I$ is zero, and then replace d by d'. In other words, we can assume that

$$\begin{bmatrix} [b, d] \\ [c, d] \end{bmatrix} = A \begin{bmatrix} [a, b] \\ [a, c] \end{bmatrix},$$

where A has zero trace, so that the characteristic polynomial of A is $x^2 - \mu$ for some μ. By scaling a we can replace μ by $k^2\mu$ for any k. So we may take $\mu = 0, 1$ or ω

(where ω is a generator of the multiplicative group of non-zero elements in $\mathbb{Z}_p$). Now consider the eœect of replacing b, c by b', c' where

$$\begin{bmatrix} b' \\ c' \end{bmatrix} = P \begin{bmatrix} b \\ c \end{bmatrix}$$

for some non-singular matrix P. Then

$$\begin{bmatrix} [b', d] \\ [c', d] \end{bmatrix} = PAP^{-1} \begin{bmatrix} [a, b'] \\ [a, c'] \end{bmatrix}.$$

So we can assume that

$$\begin{bmatrix} [b, d] \\ [c, d] \end{bmatrix} = A \begin{bmatrix} [a, b] \\ [a, c] \end{bmatrix}$$

where A is one of

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & \omega \\ 1 & 0 \end{bmatrix}.$$

If $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, then d is central which is dealt with above. If $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ then d has breadth 1, which is also dealt with above. If $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ then we have $[b, d] = [a, b]$, $[c, d] = -[a, c]$. But then if we let $d' = a + d$ we have $[a, d'] = [b, d'] = 0$, so that d' has breadth 1. Finally, consider the case when $A = \begin{bmatrix} 0 & \omega \\ 1 & 0 \end{bmatrix}$. Then we have the algebra [DW.38]

$$\langle a, b, c, d \mid [a, d] = 0,\ [b, d] = \omega[a, c],\ [c, d] = [a, b],\ \text{class } 2\rangle. \tag{7.20}$$

So far, we have dealt with the situation when L has an element of breadth 1, or an element of breadth 2. We show that this covers all situations, by showing that L always has some element of breadth 1 or 2.

So suppose that L is generated by a, b, c, d, and suppose that every non-zero element in the linear span of a, b, c, d has bread 3. Then L^2 is spanned by $[a, b]$, $[a, c]$, $[a, d]$. We write

$$\begin{bmatrix} [b, d] \\ [c, d] \end{bmatrix} = A \begin{bmatrix} [a, b] \\ [a, c] \end{bmatrix} + \begin{bmatrix} \alpha[a, d] \\ \beta[a, d] \end{bmatrix}$$

for some 2×2 matrix A and some α, β. Replacing b by $b - \alpha a$ and replacing c by $c - \beta a$ we have

$$\begin{bmatrix} [b, d] \\ [c, d] \end{bmatrix} = A \begin{bmatrix} [a, b] \\ [a, c] \end{bmatrix}.$$

If we set $d' = d + \lambda a$ then we have

$$\begin{bmatrix} [b, d'] \\ [c, d'] \end{bmatrix} = (A - \lambda I) \begin{bmatrix} [a, b] \\ [a, c] \end{bmatrix},$$

so we may assume that A has trace zero, and characteristic polynomial $x^2 - \mu$ for some μ. By scaling a we can replace μ by $k^2\mu$ for any k. So we may take $\mu = 0, 1$ or ω. As above, we consider the effect of replacing b, c by b', c' where

$$\begin{bmatrix} b' \\ c' \end{bmatrix} = P \begin{bmatrix} b \\ c \end{bmatrix}$$

for some non-singular matrix P. Then

$$\begin{bmatrix} [b', d] \\ [c', d] \end{bmatrix} = PAP^{-1} \begin{bmatrix} [a, b'] \\ [a, c'] \end{bmatrix}.$$

So we can assume that

$$\begin{bmatrix} [b, d] \\ [c, d] \end{bmatrix} = A \begin{bmatrix} [a, b] \\ [a, c] \end{bmatrix}$$

where A is one of

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & \omega \\ 1 & 0 \end{bmatrix}.$$

In the first two case $[c, d] = 0$ so that c and d have breadth at most 2. In the third case $[b, a + d] = 0$, so that b has breadth at most 2. So suppose that $A = \begin{bmatrix} 0 & \omega \\ 1 & 0 \end{bmatrix}$. Then $[b, d] = \omega[a, c]$ and $[c, d] = [a, b]$. We suppose that

$$[b, c] = \alpha[a, b] + \beta[a, c] + \gamma[a, d].$$

Note that $\gamma \neq 0$, for if $\gamma = 0$ then $[b - \beta a, c + \alpha a] = 0$, and $b - \beta a$ has breadth at most 2. We show that we can choose λ, μ so that $a + \lambda b + \mu c$ has breadth at most 2. It is straightforward to show that $[a + \lambda b + \mu c, L]$ is spanned by

$$(1 - \mu\alpha)[a, b] - \mu\beta[a, c] - \mu\gamma[a, d],$$

$$\lambda\alpha[a, b] + (1 + \lambda\beta)[a, c] + \lambda\gamma[a, d],$$

$$\mu[a, b] + \lambda\omega[a, c] + [a, d].$$

It follows that $a + \lambda b + \mu c$ has breadth at most 2 provided

$$\det \begin{bmatrix} 1 - \mu\alpha & -\mu\beta & -\mu\gamma \\ \lambda\alpha & 1 + \lambda\beta & \lambda\gamma \\ \mu & \lambda\omega & 1 \end{bmatrix} = 0.$$

So we need to find λ, μ such that $1 + \lambda\beta - \lambda^2\gamma\omega - \mu\alpha + \mu^2\gamma = 0$. We rewrite this equation as

$$(\mu - \frac{\alpha}{2\gamma})^2 = (\lambda - \frac{\beta}{2\omega\gamma})^2\omega + \frac{\alpha^2}{4\gamma^2} - \frac{\beta^2}{4\omega\gamma^2} - \frac{1}{\gamma}.$$

Suppose we øx a value of λ. Then clearly we can ønd a value of μ satsifying this equation provided

$$(\lambda - \frac{\beta}{2\omega\gamma})^2\omega + \frac{\alpha^2}{4\gamma^2} - \frac{\beta^2}{4\omega\gamma^2} - \frac{1}{\gamma}$$

is a square in $\mathbb{Z}_p$. Now half the non-zero elements of $\mathbb{Z}_p$ are squares, and half are not squares. Furthermore 0 is a square. So $(p+1)/2$ elements in $\mathbb{Z}_p$ are squares, and $(p-1)/2$ are not squares. As λ ranges over $\mathbb{Z}_p$,

$$(\lambda - \frac{\beta}{2\omega\gamma})^2\omega + \frac{\alpha^2}{4\gamma^2} - \frac{\beta^2}{4\omega\gamma^2} - \frac{1}{\gamma}$$

takes $(p+1)/2$ diœerent values, and so it there is some λ for which its value is a square. Thus we can ønd λ, μ so that $a + \lambda b + \mu c$ has breadth at most 2.

It remains to show that the algebras 7.15˘7.20 are distinct. Algebra 7.15 is the only one of the six in which the centre strictly contains the derived algebra. The algebras 7.16˘7.20 have the following numbers of elements of breadth 1: $2p^3(p-1)$, $p^3(p^3-1)$, $p^3(p-1)$, 0, 0. We distinguish between 7.19 and 7.20 by considering elements of breadth 2: algebra 7.19 has $p^3(p+1)^2(p-1)$ elements of breadth 2, and algebra 7.20 has $p^3(p^2-1)$ elements of breadth 2.

5.4.2 case 2

Now suppose that L is an immediate descendant of 5.2. Then L is generated by a, b, c, d, L^2 is generated by $[b, a]$ modulo L^3, and L^3 is spanned by $[b, a, a]$, $[b, a, b]$. Note that $[b, a, a]$ and $[b, a, b]$ are linearly independent, since L^3 has dimension 2. The commutators $[c, a]$, $[c, b]$, $[d, a]$, $[d, b]$, $[d, c]$ must all be linear combinations of $[b, a, a]$ and $[b, a, b]$.

We deal separately with the cases when $[d, c] = 0$ and the cases when $[d, c] \neq 0$.

First, consider the case when $[d, c] = 0$.

If some non-trivial linear combination of c and d is central then there is no loss in generality in assuming that d is central, so that $L = B \oplus \langle d \rangle$, where B is an imediate descendant of 4.2 of dimension 6. So $L = B \oplus \langle d \rangle$, where B is one of 6.12 ˘ 6.15. this gives [DW.5, DW.14, DW.16, DW.19]

$$\langle a, b \,|\, \text{class } 3 \rangle \oplus \langle c \rangle \oplus \langle d \rangle, \tag{7.21}$$

$$\langle a, b, c \,|\, [c, a] = 0,\ [c, b] = [b, a, a],\ \text{class } 3 \rangle \oplus \langle d \rangle, \tag{7.22}$$

$$\langle a, b, c \,|\, [c, a] = 0,\ [c, b] = [b, a, b],\ \text{class } 3 \rangle \oplus \langle d \rangle, \tag{7.23}$$

$$\langle a, b, c \,|\, [c, a] = [b, a, b],\ [c, b] = \omega[b, a, a],\ \text{class } 3 \rangle \oplus \langle d \rangle. \tag{7.24}$$

If no non-trivial linear combination of c and d is central then $\langle a, b, c \rangle$ and $\langle a, b, d \rangle$ are each isomorphic to one of 6.13 ˘ 6.15. We show that it is always possible to choose

a, b, c, d so that $\langle a, b, d\rangle$ is isomorphic to 6.14. this means that we can assume that $[d, a] = 0$, $[d, b] = [b, a, b]$.

First we analyze the conditions under which $\langle a, b, d\rangle$ is isomorphic to 6.14. We can write

$$\begin{bmatrix} [d, a] \\ [d, b] \end{bmatrix} = B \begin{bmatrix} [b, a, a] \\ [b, a, b] \end{bmatrix}$$

for some 2×2 matrix B. If A is any non-singular 2×2 matrix over $\mathbb{Z}_p$ and

$$\begin{bmatrix} a' \\ b' \end{bmatrix} = A \begin{bmatrix} a \\ b \end{bmatrix}$$

then

$$\begin{bmatrix} [d, a'] \\ [d, b'] \end{bmatrix} = \frac{1}{\det A} ABA^{-1} \begin{bmatrix} [b', a', a'] \\ [b', a', b'] \end{bmatrix}.$$

If B is diagonalisable then we can replace a, b by suitable a', b' so that

$$\begin{bmatrix} [d, a] \\ [d, b] \end{bmatrix} = \begin{bmatrix} \lambda[b, a, a] \\ \mu[b, a, b] \end{bmatrix}$$

for some λ, μ. Note that if $\lambda = \mu$ then $d - \lambda[b, a]$ is central, and we are back in an earlier case. So we assume that $\lambda \neq \mu$. Then replacing d by $d - \lambda[b, a]$ and scaling we have

$$\begin{bmatrix} [d, a] \\ [d, b] \end{bmatrix} = \begin{bmatrix} 0 \\ [b, a, b] \end{bmatrix}.$$

So we need to show that there always exists an element d in the linear span of c and d for which the associated matrix B is diagonalisable.

Now, as we stated earlier, the subalgebra $\langle a, b, c\rangle$ is isomorphic to 6.13, 6.14 or 6.15. If $\langle a, b, c\rangle$ is isomorphic to 6.14 then we can take $d = c$. So we may assume that $\langle a, b, c\rangle$ is isomorphic to 6.13 or 6.15.

First suppose that $\langle a, b, c\rangle$ is isomorphic to 6.13. Then we may suppose that $[c, a] = [b, a, b]$, $[c, b] = 0$. Let

$$\begin{bmatrix} [d, a] \\ [d, b] \end{bmatrix} = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \begin{bmatrix} [b, a, a] \\ [b, a, b] \end{bmatrix}$$

Replacing d by $d - \beta c$ we have

$$\begin{bmatrix} [d, a] \\ [d, b] \end{bmatrix} = \begin{bmatrix} \alpha & 0 \\ \gamma & \delta \end{bmatrix} \begin{bmatrix} [b, a, a] \\ [b, a, b] \end{bmatrix}.$$

Now the matrix $\begin{bmatrix} \alpha & 0 \\ \gamma & \delta \end{bmatrix}$ is diagonalizable unless $\alpha = \delta$ and $\gamma \neq 0$. So we are done unless $\alpha = \delta$. In this case replace d by $d - \alpha[b, a]$ and we have the matrix $\begin{bmatrix} 0 & 0 \\ \gamma & 0 \end{bmatrix}$.

If $\gamma = 0$ then d is central, and we are back in an earlier case. If $\gamma \neq 0$ then scaling d we may suppose that $\gamma = 1$. But then

$$\begin{bmatrix} [c+d,a] \\ [c+d,b] \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} [b,a,a] \\ [b,a,b] \end{bmatrix},$$

and we are done since $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ is diagonalizable.

Next suppose that $\langle a,b,c\rangle$ is isomorphic to 6.15. Then we may choose a,b,c so that the matrix associated with a is $\begin{bmatrix} 0 & 1 \\ \omega & 0 \end{bmatrix}$. If the matrix associated with d is $\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$, then we replace d by $d - \beta a$, and proceed as above.

So, in every case we may assume that $[d,a] = 0$, $[d,b] = [b,a,b]$. Let

$$\begin{bmatrix} [c,a] \\ [c,b] \end{bmatrix} = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \begin{bmatrix} [b,a,a] \\ [b,a,b] \end{bmatrix}.$$

Replacing c by $c + (\alpha - \delta)b - \alpha[b,a]$ we have

$$\begin{bmatrix} [c,a] \\ [c,b] \end{bmatrix} = \begin{bmatrix} 0 & \beta \\ \gamma & 0 \end{bmatrix} \begin{bmatrix} [b,a,a] \\ [b,a,b] \end{bmatrix}.$$

Note that we can assume that at least one of β, γ is non-zero, or we are back in an earlier case. If $\beta = 0$ then we can scale c so that $\gamma = 1$, and then we have [DW.56]

$$\langle a,b,c,d \mid [c,a] = [d,a] = [d,c] = 0,\ [c,b] = [b,a,a],\ [d,b] = [b,a,b], \text{ class } 3\rangle. \quad (7.25)$$

If $\gamma = 0$ then we can scale c so that $\beta = 1$. If we then replace d by $-d + [b,a]$ we have

$$\begin{aligned} [c,b] &= [d,b] = 0, \\ [c,a] &= [b,a,b], \\ [d,a] &= [b,a,a] \end{aligned}$$

and we have 7.25

again. So we may assume that β, γ are both non-zero, and then by scaling we have

$$\begin{aligned} [c,a] &= [b,a,b], \\ [c,b] &= \lambda[b,a,a], \end{aligned}$$

where $\lambda = 1$ or ω. This gives [DW.57, DW.58]

$$\begin{aligned} \langle a,b,c,d \mid [c,a] &= [d,b] = [b,a,b],\ [c,b] = \lambda[b,a,a],\ [d,a] = [d,c] = 0, \quad (7.26) \\ (\lambda &= 1,\omega), \text{ class } 3\rangle. \end{aligned}$$

Next we consider the case when $[d, c] \neq 0$. Clearly we may suppose that $[d, c] = [b, a, a]$. Replacing b by $b + \alpha c + \beta d$ we may suppose that $[c, b]$, $[d, b]$ lie in the span of $[b, a, b]$. But then replacing c by $c + \gamma[b, a]$ we may suppose that $[c, b] = 0$. Similarly we may suppose that $[d, b] = 0$. If $[c, a] = [d, a] = 0$ then we have [DW.54]

$$\langle a, b \mid \text{class } 3\rangle \oplus_{[b,a,a]=[d,c]} \langle c, d \mid \text{class } 2\rangle. \tag{7.27}$$

If $[c, a]$ and $[d, a]$ span a space of dimension 1 then we may suppose that $[d, a] = 0$, and that $[c, a] = \alpha[b, a, a] + \beta[b, a, b]$ for some α, β not both zero. Replacing c by $c - \alpha[b, a]$ we have

$$\begin{aligned} [c, a] &= \beta[b, a, b], \\ [c, b] &= -\alpha[b, a, b], \\ [d, a] &= [d, b] = 0, \\ [d, c] &= [b, a, a]. \end{aligned}$$

If $\beta = 0$ we let $a' = a + \alpha d$, $b' = b$, $c' = c + \alpha[b, a]$, $d' = d$. Then a', b', c', d' satisfy the relations of 7.27, so this case is covered above. If $\beta \neq 0$ then we can replace b by $b + \alpha\beta^{-1}a$ so that $[c, b] = 0$. So we can assume that $\alpha = 0$ and by scaling c and d we obtain [DW.55]

$$\langle a, b, c, d \mid [c, b] = [d, a] = [d, b] = 0,\ [c, a] = [b, a, b],\ [d, c] = [b, a, a], \text{ class } 3\rangle. \tag{7.28}$$

On the other hand, if $[c, a]$ and $[d, a]$ span a space of dimension 2 then we may suppose that $[c, a] = \lambda[b, a, b]$, $[d, a] = \mu[b, a, a]$ for some non-zero λ, μ. But then substituting $a - \mu c$ for a we have $[d, a] = 0$, and we are back to the previous case.

We now show that the algebras 7.21˘7.28 are all distinct. In all the algebras the centralizer, C, of L^2 is spanned by c, d, $[b, a]$, $[b, a, a]$, $[b, a, b]$. This centralizer C is non-abelian in 7.27 and 7.28, and abelian in the others. In 7.21˘7.24 $\langle d\rangle$ is a direct summand, and when we quotient out this direct summand we get non-isomorphic algebras. None of the remaining algebras have an abelian direct summand. To distinguish 7.27 and 7.28 we count the number of elements of breadth 2 in the centralizer of L^2. Consider an element

$$\alpha c + \beta d + \gamma[b, a] + \delta[b, a, a] + \varepsilon[b, a, b].$$

In 7.27 this has breadth 2 if and only if $\gamma \neq 0$. This gives $(p - 1)p^4$ elements of bread 2. But in 7.28 this element has breadth 2 if $\alpha \neq 0$ or $\gamma \neq 0$, so in this case there are $(p - 1)(p + 1)p^3$ elements of breadth 2. So the two algebras are distinct. In 7.25, a has breadth 2, but in both algebras 7.26 every element outside C has breadth 3.

So it only remains to show that the values $\lambda = 1, \omega$ in 7.26 give diœerent algebras. To this end we let L be 7.26 with $\lambda = 1$, and we consider all possible generating sets a', b', c', d' for L which satsify the same relations as a, b, c, d (except possibly with a diœerent value of λ).

First consider the situation when $a' = \alpha a$, $b' = \beta b$, $c' = \gamma c$, $d' = \delta d$. Then we need

$$\alpha\gamma = \beta\delta = \alpha\beta^2,$$

so $\gamma = \beta^2$, $\delta = \alpha\beta$. This gives a new value of λ equal to $\alpha^{-2}\beta^2$. So by scaling we can only obtain values of λ which are squares.

Now consider the general case. Since $C = \langle c, d\rangle + L^2$ is the inverse image in L of the centre of L/L^3 it is clear that $c', d' \in C$. Let

$$\begin{aligned} a' &= \alpha a + \beta b + u, \\ b' &= \gamma a + \delta b + v, \\ c' &= \varepsilon c + \zeta d + \eta [b, a] + x, \\ d' &= \mu c + \nu d + \xi [b, a] + y, \end{aligned}$$

with $u, v \in C$, $x, y \in L^3$. Since d has breadth 1, d' must have breadth 1, and this implies that $\mu^2 - \xi^2 - \nu\xi = 0$. So either $\mu = \xi = 0$, or $\xi \neq 0$ and $\nu = \mu^2\xi^{-1} - \xi$.

First consider the case when $\mu = \xi = 0$. Then the relation $[d', a'] = 0$ implies that $\beta = 0$. We also have

$$\begin{aligned} [d', b'] &= \delta\nu[d, b] = \delta\nu[b, a, b], \\ [b', a'] &= (\alpha\delta - \beta\gamma)[b, a] \bmod L^3, \\ [b', a', b'] &= (\alpha\delta - \beta\gamma)(\gamma[b, a, a] + \delta[b, a, b]). \end{aligned}$$

So the relation $[d', b'] = [b', a', b']$ implies that $\gamma = 0$ and $\nu = \alpha\delta - \beta\gamma$. So

$$\begin{aligned} [c', a'] &= \alpha\varepsilon[c, a] + \alpha\eta[b, a, a], \\ [c', b'] &= \delta\varepsilon[c, b] + \delta\zeta[d, b] + \delta\eta[b, a, b], \\ [b', a', a'] &= (\alpha\delta - \beta\gamma)\alpha[b, a, a]. \end{aligned}$$

So the relation $[c', a'] = [b', a', b']$ implies that $\eta = 0$ and $\alpha\varepsilon = (\alpha\delta - \beta\gamma)\delta$. And the relation $[c', b'] = \lambda[b', a', a']$ implies that $\zeta = 0$, and that λ equals

$$\frac{\delta\varepsilon}{(\alpha\delta - \beta\gamma)\alpha} = \frac{\delta^2}{\alpha^2}.$$

So again we only have square values for λ.

Next consider the case when $\xi \neq 0$ and $\nu = \mu^2\xi^{-1} - \xi$. Scaling d' we may suppose that $\xi = 1$.Then

$$\begin{aligned} [d', a'] &= \alpha\mu[c, a] + \beta\mu[c, b] + (\mu^2 - 1)\beta[d, b] + \alpha[b, a, a] + \beta[b, a, b], \\ [d', b'] &= \gamma\mu[c, a] + \delta\mu[c, b] + (\mu^2 - 1)\delta[d, b] + \gamma[b, a, a] + \delta[b, a, b], \\ [b', a', a'] &= (\alpha\delta - \beta\gamma)(\alpha[b, a, a] + \beta[b, a, b]), \\ [b', a', b'] &= (\alpha\delta - \beta\gamma)(\gamma[b, a, a] + \delta[b, a, b]). \end{aligned}$$

So the relations $[d', a'] = 0$ and $[d', b'] = [b', a', b']$ imply that

$$\begin{aligned} \beta\mu + \alpha &= 0, \\ \alpha\mu + \beta\mu^2 &= 0, \\ \delta\mu + \gamma &= (\alpha\delta - \beta\gamma)\gamma, \\ \gamma\mu + \delta\mu^2 &= (\alpha\delta - \beta\gamma)\delta. \end{aligned}$$

The ørst equation above implies that $\alpha = -\beta\mu$, and then the fact that we must have $\alpha\delta - \beta\gamma \neq 0$ implies that $\beta \neq 0$. Scaling a' we may suppose that $\beta = 1$. So

$$\begin{aligned} \alpha &= -\mu, \\ \beta &= 1, \\ \delta\mu + \gamma &= -(\delta\mu + \gamma)\gamma, \\ \gamma\mu + \delta\mu^2 &= -(\delta\mu + \gamma)\delta. \end{aligned}$$

Since $-\delta\mu - \gamma = \alpha\delta - \beta\gamma \neq 0$, These equations give $\gamma = -1$, $\delta = -\mu$. So we have

$$\begin{aligned} a' &= -\mu a + b + u, \\ b' &= -a - \mu b + v, \\ c' &= \varepsilon c + \zeta d + \eta[b, a] + x, \\ d' &= \mu c + (\mu^2 - 1)d + [b, a] + y, \\ [b', a', a'] &= (\mu^2 + 1)(-\mu[b, a, a] + [b, a, b]), \\ [b', a', b'] &= (\mu^2 + 1)(-[b, a, a] - \mu[b, a, b]). \end{aligned}$$

So

$$\begin{aligned} [c', a'] &= -\varepsilon\mu[c, a] - \eta\mu[b, a, a] + \varepsilon[c, b] + \zeta[d, b] + \eta[b, a, b] \\ &= (-\eta\mu + \varepsilon)[b, a, a] + (-\varepsilon\mu + \zeta + \eta)[b, a, b], \\ [c', b'] &= -\varepsilon[c, a] - \eta[b, a, a] - \varepsilon\mu[c, b] - \zeta\mu[d, b] - \eta\mu[b, a, b] \\ &= (-\eta - \varepsilon\mu)[b, a, a] + (-\varepsilon - \zeta\mu - \eta\mu)[b, a, b]. \end{aligned}$$

Since $[c', a'] = [b', a', b']$ we have

$$\begin{aligned} -\eta\mu + \varepsilon &= -(\mu^2 + 1), \\ -\varepsilon\mu + \zeta + \eta &= -(\mu^2 + 1)\mu, \end{aligned}$$

and since $[c', b']$ is a scalar multiple of $[b', a', a']$ we have

$$(\varepsilon + \zeta\mu + \eta\mu)\mu + \eta + \varepsilon\mu = 0,$$

with the new value of λ given by

$$\lambda' = \frac{-\varepsilon - \zeta\mu - \eta\mu}{\mu^2 + 1}.$$

These three equations give

$$\begin{aligned}\varepsilon &= \mu^2 - 1,\\ \zeta &= -4\mu,\\ \eta &= 2\mu,\end{aligned}$$

and so $\lambda' = 1$. So the two values $\lambda = 1, \omega$ in 7.26 give dϊeerent algebras.

5.4.3 case 3

We now consider the case when L is an immediate descendant of 6.4. Then L is generated by a, b, c, d, L^2 is generated by $[b, a]$ and $[c, a]$ modulo L^3, and L^3 is generated by $[b, a, a]$, $[b, a, b]$, $[b, a, c] = [c, a, b]$, $[c, a, a]$, $[c, a, c]$. The subalgebra generated by a, b, c has dimension 6, and so is one of 6.16 ˘ 6.21.

First suppose that $\langle a, b, c\rangle$ is 6.16. Then L^3 is spanned by $[b, a, b]$ and

$$[c, b] = [b, a, a] = [c, a, a] = [b, a, c] = [c, a, c] = 0.$$

The commutators $[d, a]$, $[d, b]$, $[d, c]$ are all linear multiples of $[b, a, b]$. If $[d, b] = \alpha[b, a, b]$ then replacing d by $d - \alpha[b, a]$ we may assume that $[d, b] = 0$. One possibility is $[d, a] = [d, c] = 0$ and this give [DW.15]

$$\langle a, b, c \,|\, [c, b] = [b, a, a] = [c, a, a] = [b, a, c] = [c, a, c] = 0, \text{ class } 3\rangle \oplus \langle d\rangle. \tag{7.29}$$

If $[d, c] \neq 0$ then we can scale d so that $[d, c] = [b, a, b]$, and then replacing a by $a + \lambda c$ for some λ we may assume that $[d, a] = 0$. This gives [DW.47]

$$\begin{aligned}\langle a, b, c, d \,|\, [c, b] &= [d, a] = [d, b] = [b, a, a] = [c, a, a] = [b, a, c] = [c, a, c] = 0,\\ [d, c] &= [b, a, b], \text{ class } 3\rangle.\end{aligned} \tag{7.30}$$

On the other hand, if $[d, c] = 0$ but $[d, a] \neq 0$ then we can scale d so that $[d, a] = [b, a, b]$ [DW.48]

$$\begin{aligned}\langle a, b, c, d \,|\, [c, b] &= [d, b] = [d, c] = [b, a, a] = [c, a, a] = [b, a, c] = [c, a, c] = 0,\\ [d, a] &= [b, a, b], \text{ class } 3\rangle.\end{aligned} \tag{7.31}$$

Next suppose that $\langle a, b, c\rangle$ is 6.17. Then L^3 is spanned by $[b, a, b]$ and

$$\begin{aligned}[c, b] &= [b, a, a] = [b, a, c] = [c, a, c] = 0,\\ [c, a, a] &= [b, a, b].\end{aligned}$$

We get three algebras, just as above [DW.26, DW.114]

$$\langle a, b, c \,|\, [c, b] = [b, a, a] = [b, a, c] = [c, a, c] = 0, [c, a, a] = [b, a, b], \text{ class } 3\rangle \oplus \langle d\rangle, \tag{7.32}$$

$$\begin{aligned} \langle a, b, c, d \mid [c, b] &= [d, a] = [d, b] = [b, a, a] = [b, a, c] = [c, a, c] = 0, \\ [d, c] &= [c, a, a] = [b, a, b], \text{ class } 3\rangle, \end{aligned} \quad (7.33)$$

and

$$\begin{aligned} \langle a, b, c, d \mid [c, b] &= [d, b] = [d, c] = [b, a, a] = [b, a, c] = [c, a, c] = 0, \\ [d, a] &= [c, a, a] = [b, a, b], \text{ class } 3\rangle. \end{aligned}$$

However if we let $d' = d - [c, a]$ in this last algebra then a, b, c, d' satisfy the relations of 7.32, so this algebra is already covered.

Now we consider the case when $\langle a, b, c\rangle$ is 6.18 or 6.19. Again L^3 is spanned by $[b, a, b]$ and

$$\begin{aligned} [c, b] &= [b, a, a] = [b, a, c] = [c, a, a] = [c, a, b] = 0, \\ [c, a, c] &= \xi[b, a, b] \end{aligned}$$

where $\xi = 1$ or ω. Replacing d by $d + \alpha[b, a] + \beta[c, a]$ for suitable α, β we may suppose that $[d, b] = [d, c] = 0$, and scaling d we may suppose that $[d, a] = 0$ or $[b, a, b]$. This gives [DW.24, DW.25, DW.115, DW.116]

$$\langle a, b, c \mid [c, b] = [b, a, a] = [b, a, c] = [c, a, a] = 0,\ [c, a, c] = [b, a, b], \text{ class } 3\rangle \oplus \langle d\rangle, \quad (7.34)$$

$$\langle a, b, c \mid [c, b] = [b, a, a] = [b, a, c] = [c, a, a] = 0,\ [c, a, c] = \omega[b, a, b], \text{ class } 3\rangle \oplus \langle d\rangle, \quad (7.35)$$

$$\begin{aligned} \langle a, b, c, d \mid [c, b] &= [d, b] = [d, c] = [b, a, a] = [b, a, c] = [c, a, a] = 0, \\ [d, a] &= [c, a, c] = [b, a, b], \text{ class } 3\rangle, \end{aligned} \quad (7.36)$$

$$\begin{aligned} \langle a, b, c, d \mid [c, b] &= [d, b] = [d, c] = [b, a, a] = [b, a, c] = [c, a, a] = 0, \\ [d, a] &= [b, a, b],\ [c, a, c] = \omega[b, a, b], \text{ class } 3\rangle, \end{aligned} \quad (7.37)$$

(Note that the group DW.117 is isomorphic to DW.115 or DW.116 depending on whether or not -1 is a square in $\mathbb{Z}_p$. To see this, let a, b, c, d, e, f, g be the generators in the presentation for DW.117, and let $a' = ab^{-1}c$, $b' = bc$, $c' = bc^{-1}$, $f' = f^2d^{-1}e$. Then let $d' = [b', a']$, $e' = [c', a']$, $g' = [d', b']$. It is straightforward to check that all commutators in $a', b', c', d', e', f', g'$ are trivial except for

$$\begin{aligned} [b', a'] &= d', \\ [c', a'] &= e', \\ [d', b'] &= g', \\ [e', a'] &= g', \\ [e', c'] &= g'^{-1}, \\ [f', c'] &= g'. \end{aligned}$$

So DW.117 is isomorphic to DW.115 if -1 is a square, and isomorphic to DW.116 if -1 is not a square.)

If $\langle a, b, c\rangle$ is 6.20 then L^3 is spanned by $[b, a, a]$ and

$$[c, b] = [b, a, b] = [b, a, c] = [c, a, a] = [c, a, b] = [c, a, c] = 0.$$

Replacing d by $d + \alpha[b, a]$ for suitable α we may assume that $[d, a] = 0$. If $[d, c] = 0$ then we can assume that $[d, b] = 0$ or $[b, a, a]$. And if $[d, c] \neq 0$ we can assume that $[d, c] = [b, a, a]$, and replacing b by $b+\beta c$ for suitable β we may assume that $[d, b] = 0$. So we have three algebras [DW.17, DW.50, DW.49]

$$\langle a, b, c \,|\, [c, b] = [b, a, b] = [b, a, c] = [c, a, a] = [c, a, c] = 0, \text{ class } 3\rangle \oplus \langle d\rangle, \qquad (7.38)$$

$$\begin{aligned} \langle a, b, c, d \,|\, [c, b] &= [d, a] = [d, c] = [b, a, b] = [b, a, c] = [c, a, a] = [c, a, c] = 0, \quad (7.39)\\ [d, b] &= [b, a, a], \text{ class } 3\rangle, \end{aligned}$$

$$\begin{aligned} \langle a, b, c, d \,|\, [c, b] &= [d, a] = [d, b] = [b, a, b] = [b, a, c] = [c, a, a] = [c, a, c] = 0, \quad (7.40)\\ [d, c] &= [b, a, a], \text{ class } 3\rangle. \end{aligned}$$

Finally consider the case when $\langle a, b, c\rangle$ is 6.21. Again L^3 is spanned by $[b, a, a]$, but now we have

$$[b, a, b] = [b, a, c] = [c, a, a] = [c, a, b] = [c, a, c] = 0, \; [c, b] = [b, a, a].$$

Once again we get three algebras [DW.18 and two others]

$$\langle a, b, c \,|\, [b, a, b] = [b, a, c] = [c, a, a] = [c, a, c] = 0, \; [c, b] = [b, a, a], \text{ class } 3\rangle \oplus \langle d\rangle, \qquad (7.41)$$

$$\begin{aligned} \langle a, b, c, d \,|\, [d, a] &= [d, c] = [b, a, b] = [b, a, c] = [c, a, a] = [c, a, c] = 0,\\ [c, b] &= [d, b] = [b, a, a], \text{ class } 3\rangle, \end{aligned}$$

$$\begin{aligned} \langle a, b, c, d \,|\, [d, a] &= [d, b] = [b, a, b] = [b, a, c] = [c, a, a] = [c, a, c] = 0,\\ [c, b] &= [d, c] = [b, a, a], \text{ class } 3\rangle. \end{aligned}$$

But if we replace c by $c - d$ in the second of these algebras then we get 7.39 again, and if we replace b by $b + d$ in the third of these algebras we get 7.40 again. So there is only one new algebra here.

We need to show that the algebras 7.29 ˘ 7.41 are distinct. First note that d is central in the algebras 7.29, 7.32, 7.34, 7.35, 7.38, 7.41, whereas the centres of the remaining algebras is contained in L^2. And for these six algebras the quotients $L/\langle d\rangle$ are the six diœerent algebras 6.16 ˘ 6.21. Next note that $[c, a]$ is central in 7.30, 7.31, 7.39, 7.40, so that these algebras have centres of dimension 2, whereas the centres of 7.33, 7.36, 7.37 have dimension 1. So it remains to distinguish 7.30, 7.31, 7.39, 7.40 from each other, and to distinguish 7.33, 7.36, 7.37 from each other.

We note that in all these algebras the centralizer of L modulo L^3 is the subalgebra $D = \langle d\rangle + L^2$, and hence that D is characteristic. Also if we let $B = \langle b, c, d\rangle + L^2$ then B/L^3 is the only øve dimensional abelian subalgebra of L/L^3. So B is also characteristic.

First we distinguish between 7.30 from 7.31, 7.39, 7.40 by counting the number of elements of breadth 3 in each algebra. In all these algebras every element in the subalgebra B has breadth at most 2. In 7.31, 7.39, 7.40 every element outside B has breadth 3, but in 7.30 the element a only has breadth 2. So 7.30 is not isomorphic to the other three algebras. Next note that B centralizes L^2 in 7.39 and 7.40, but not in 7.31. To diœerentiate between 7.39 and 7.40 we note that in 7.39 a, c, d generate a subalgebra C with the following properties.

1. $\dim(C) = 4$,
2. $\dim(C \cap L^2) = 1$,
3. $\dim(C \cap L^3) = 0$,
4. C is generated by 3 elements.

We show that 7.40 does contain a subalgebra C with these properties. Suppose, for a contradiction, that we had such a subalgebra C in 7.40. Note that $C \cap L^2 = C^2$. So C cannot be a subalgebra of B, since $B^2 = L^3$. So C contains an element $a' = a + e$ for some $e \in B$. If C contained an element $b' = b + f$ for some $f \in \langle c, d\rangle + L^2$, then (using the fact that $[c, a]$ is central) we see that C would contain the element $[b', a', a'] = [b, a, a]$. This would contradict property 3 above. So C must be generated by a', c', d' with $c' = c + g$, $d' = d + h$ for some $g, h \in L^2$. But this implies that C contains $[d', c'] = [b, a, a]$, a contradiction.

Now consider 7.33, 7.36 and 7.37. In 7.36 and 7.37 $\langle b, c, d\rangle$ is an abelian subalgebra of B of dimension 3 with trivial intersection with L^2. The algebra 7.33 has no such subalgebra. Finally, in 7.36 and 7.37 a, b, c span a complement to the characteristic subalgebra $D = \langle d\rangle + L^2$ and if a', b', c' are any other three elements spanning such a complement then $\langle a, b, c\rangle$ is isomorphic to $\langle a', b', c'\rangle$. In 7.36 $\langle a, b, c\rangle$ is isomorphic to 6.18, and in 7.37 it is isomorphic to 6.19.

5.4.4 case 4

Next consider the case when L is an immediate descendant of 6.5. Then L is generated by a, b, c, d, L^2 is generated by $[b, a]$ and $[d, c]$ modulo L^3, and L^3 is generated by $[b, a, a]$, $[b, a, b]$, $[d, c, c]$, $[d, c, d]$. In addition the commutators $[c, a]$, $[d, a]$, $[c, b]$, $[d, b]$

all lie in L^3. We may suppose that L^3 is spanned by $[b,a,b]$ and that $[b,a,a] = [d,c,c] = 0$ and $[d,c,d] = 0$ or $[b,a,b]$.

First consider the case when $[d,c,d] = 0$, so that $[d,c]$ is central. Replacing c by $c + \alpha[b,a]$ for suitable α we may assume that $[c,b] = 0$. Similarly we may suppose that $[d,b] = 0$. Since $[c,a]$ and $[d,a]$ span a space of dimension at most 1 we may suppose that $[d,a] = 0$ and that $[c,a] = \lambda[b,a,b]$ for some λ. Scaling c we may assume that $\lambda = 0$ or 1. this gives [DW.45, DW.46]

$$\langle a,b \mid [b,a,a]=0,\ \text{class } 3\rangle \oplus \langle c,d \mid \text{class } 2\rangle, \tag{7.44}$$

$$\begin{aligned} \langle a,b,c,d \mid [d,a] &= [c,b]=[d,b]=[b,a,a]=[d,c,c]=[d,c,d]=0, \\ [c,a] &= [b,a,b],\ \text{class } 3\rangle. \end{aligned} \tag{7.45}$$

Next consider the case when $[d,c,d] = [b,a,b]$. As above, we may suppose that $[c,b] = [d,b] = 0$. And replacing a by $a + \beta[d,c]$ for suitable β we may assume that $[d,a] = 0$. We have $[c,a] = \lambda[b,a,b]$ for some λ, and by scaling a and c we may assume that $\lambda = 0$ or 1. This gives [DW.131, DW.132]

$$\langle a,b \mid [b,a,a]=0,\ \text{class } 3\rangle \oplus_{[b,a,b]=[d,c,d]} \langle c,d \mid [d,c,c]=0,\ \text{class } 3\rangle, \tag{7.46}$$

$$\begin{aligned} \langle a,b,c,d \mid [d,a] &= [c,b]=[d,b]=[b,a,a]=[d,c,c]=0, \\ [c,a] &= [b,a,b]=[d,c,d],\ \text{class } 3\rangle. \end{aligned} \tag{7.47}$$

We distinguish between these four algebras by counting the numbers of elements of breadth 1. First note that in any of the algebras an element

$$e = \alpha a + \beta b + \gamma c + \delta d + u$$

with $u \in L^2$ and $e \notin L^2$ can only have breadth 1 if $\beta = 0$, and if either $\alpha = 0$ or $\gamma = \delta = 0$. So ørst consider an element

$$\alpha a + \lambda[b,a] + \mu[d,c] + v$$

with $\alpha \neq 0$ and $v \in L^3$. This has breadth 1 in 7.44, but not in 7.45 or 7.47. In 7.46 it has breadth 1 provided $\mu = 0$. Next consider an element

$$\gamma c + \delta d + \lambda[b,a] + \mu[d,c] + v \notin L^2\ (v \in L^3).$$

This can only have breadth 1 if $\lambda = 0$. It then has breadth 1 in 7.44. In 7.45 it has breadth 1 provided $\gamma = 0$, and in 7.46 it has breadth 1 if $\delta = 0$. It cannot have breadth 1 in 7.47. Finally consider an element

$$\lambda[b,a] + \mu[d,c] + v \notin L^2\ (v \in L^3).$$

This element has breadth 1 in 7.46 and 7.47, but only has breadth 1 in 7.44 and 7.45 if $\lambda \neq 0$. So 7.44 has $2p^2(p^2-1)$ elements of breadth 1, 7.45 has $2p^2(p-1)$, 7.46 has $p(3p^2-2p-1)$, and 7.47 has $p(p^2-1)$.

5.4.5 case 5

We now consider the case when L is an immediate descendant of 6.6. Then L is generated by a, b, c, d, L^2 is generated by $[b,a]$ and $[c,a]$ modulo L^3, and L^3 is generated by $[b,a,a]$, $[b,a,b]$, $[b,a,c]$, $[b,a,d]$. The commutators $[d,a]$, $[c,b]$, $[d,c]$ all lie in L^3 and $[d,b]=[c,a]$ modulo L^3. We show that we may assume that $[b,a,a]\neq 0$.

If $[b,a,b]\neq 0$ then interchanging a and b while simultaneously interchanging c and d we have $[b,a,a]=0$. If $[b,a,a]=[b,a,b]=0$ but $[b,a,c]\neq 0$, then replacing a by $a+c$ we have $[b,a,a]\neq 0$. Finally, if $[b,a,a]=[b,a,b]=[b,a,c]=0$ then replacing b by $b+d$ we have $[b,a,b]\neq 0$, as before.

So we assume that $[b,a,a]\neq 0$, and hence that it spans L^3. Replacing b by $b+\lambda a$ for suitable λ, while simultaneously replacing c by $c-\lambda d$, gives $[b,a,b]=0$. If $[b,a,c]\neq 0$ then replacing d by $d+\mu c$ for suitable μ while simultaneously replacing a by $a-\mu b$ gives $[b,a,d]=0$. Replacing c by $c+\nu[b,a]$ for suitable ν, we may suppose that $[c,a]=[d,b]$. Similarly, we may suppose that $[d,a]=0$. Since $[c,a,b]=[b,a,c]\neq 0$, replacing c by $c+\xi[c,a]$ for suitable ξ we may suppose that $[c,b]=0$. Summarizing, L^2 is spanned modulo L^3 by $[b,a]$ and $[c,a]$, and L^3 is spanned by $[b,a,a]$. We have $[d,a]=[c,b]=0$, $[c,a]=[d,b]$ and $[d,c]=\alpha[b,a,a]$. We also have

$$\begin{aligned}[b,a,b] &= [c,a,a]=[c,a,c]=[c,a,d]=0,\\ [b,a,c] &= [c,a,b]=\beta[b,a,a],\end{aligned}$$

for some $\beta\neq 0$. Scaling c and d we may assume that $\beta=1$. Then, if $\alpha\neq 0$, replacing a by αa, c by αc and d by $\alpha^2 d$ we have $\alpha=1$. So we have two algebras [DW.133, DW.134]

$$\begin{aligned}\langle a,b,c,d\,|\,[d,a] &= [c,b]=[d,c]=0,\ [c,a]=[d,b],\\ [b,a,b] &= [b,a,d]=0,\ [b,a,c]=[b,a,a],\ \text{class } 3\rangle,\end{aligned}\tag{7.48}$$

$$\begin{aligned}\langle a,b,c,d\,|\,[d,a] &= [c,b]=0,\ [c,a]=[d,b],\\ [b,a,b] &= [b,a,d]=0, [d,c]=[b,a,c]=[b,a,a],\ \text{class } 3\rangle.\end{aligned}\tag{7.49}$$

Now suppose that $[b,a,b]=[b,a,c]=0$. We may suppose that $[b,a,d]=0$ or $[b,a,a]$. As above we may suppose that $[c,a]=[d,b]$, $[d,a]=0$. We also have $[c,b]=\alpha[b,a,a]$, $[d,c]=\beta[b,a,a]$ for some α,β. If $[b,a,d]=0$ and $[d,c]\neq 0$ then replacing b by $b+\gamma d$ for suitable γ we may suppose that $[c,b]=0$. Scaling, we may suppose that $\beta=1$. On the other hand if $[b,a,d]=[d,c]=0$ then by scaling we may suppose that $\alpha=0$ or 1. This gives [DW.51, DW.52, DW.53]

$$\begin{aligned}\langle a,b,c,d\,|\,[d,a] &= [c,b]=0,\ [c,a]=[d,b],\ [d,c]=[b,a,a],\\ [b,a,b] &= [b,a,c]=[b,a,d]=0,\ \text{class } 3\rangle,\end{aligned}\tag{7.50}$$

$$\begin{aligned}\langle a,b,c,d\,|\,[d,a] &= [c,b]=[d,c]=0,\ [c,a]=[d,b],\\ [b,a,b] &= [b,a,c]=[b,a,d]=0,\ \text{class } 3\rangle,\end{aligned}\tag{7.51}$$

$$\begin{aligned}\langle a,b,c,d \mid [d,a] &= [d,c]=0,\ [c,a]=[d,b],\ [c,b]=[b,a,a], \\ [b,a,b] &= [b,a,c]=[b,a,d]=0,\ \text{class } 3\rangle.\end{aligned} \tag{7.52}$$

Finally, if $[b,a,c]=0$, $[c,a]=[d,b]$, $[d,a]=0$, and $[b,a,d]=[b,a,a]$, then setting $a'=a$, $b'=a+b-d$, $c'=c-d$, $d'=d$ we have

$$\begin{aligned}[b',a'] &= [b,a], \\ [b',a',a'] &= [b,a,a], \\ [b',a',b'] &= 0, \\ [b',a',c'] &= -[b,a,a], \\ [d',a'] &= 0, \\ [c',a'] &= [c,a]=[d,b]=[d',b'], \\ [c',b'] &= [d,c]\in L^3, \\ [d',c'] &= [d,c]\in L^3.\end{aligned}$$

So we have 7.48 or 7.49.

We need to show that the algebras 7.48 ˘ 7.52 are distinct. To show this we count the elements of breadth 1 in the diœerent algebras. In 7.48 and 7.49 we have $[c,a,a]=0$, $[c,a,b]=[b,a,a]$. So in these two algebras everying in $L^2\backslash L^3$ has breadth 1. This gives p^3-p elements of breadth 1. In 7.49 there are no other elements of breadth 1, but in 7.48 elements $\alpha d+\beta[c,a]+\gamma[b,a,a]$ (with $\alpha\neq 0$) also has breadth 1, so 7.48 has $2p^3-p^2-p$ elements of breadth 1. In 7.50 ˘ 7.52 the centre of L is spanned by $[c,a]$ and $[b,a,a]$. So these algebras have p^3-p^2 elements of breadth 1 in the derived algebra. Algebra 7.50 has no other elements of breadth 1. In 7.51 elements

$$\alpha c+\beta d+\gamma[b,a]+\delta[c,a]+\varepsilon[b,a,a]$$

($\alpha\neq 0$ or $\beta\neq 0$, $\gamma=0$ if $\beta\neq 0$) all have breadth 1, giving $2p^4-p^3-p^2$ elements of breadth 1, in all. And in 7.52 there are p^4-p^2 elements of breadth 1.

5.4.6 case 6

Now assume that L is an immediate descendant of 6.7. Then L is generated by a,b,c,d, L^2 is generated by $[b,a]$ and $[c,a]$ modulo L^3, and L^3 is generated by $[b,a,a]$, $[b,a,b]$, $[b,a,c]$, $[b,a,d]$. The commutators $[d,a]$, $[c,b]$ lie in L^3 and $[d,b]=[c,a]$ modulo L^3, $[d,c]=\omega[b,a]$ modulo L^3. We also have

$$\begin{aligned}[c,a,a] &= -[b,a,d], \\ [c,a,b] &= [b,a,c], \\ [c,a,c] &= \omega[b,a,b], \\ [c,a,d] &= -\omega[b,a,a].\end{aligned}$$

We show that we may assume that $[b,a,a] \neq 0$. If $[b,a,b] \neq 0$, then we take $a' = b$, $b' = a$, $c' = d$, $d' = c$ and then $[b',a',a'] \neq 0$. If $[b,a,a] = [b,a,b] = 0$ then at least one of $[b,a,c]$, $[b,a,d]$ is non-zero. This means that at least one of $[d,c,c]$, $[d,c,d]$ is non-zero. But then taking $a' = d$, $b' = c$, $c' = \omega b$, $d' = \omega c$ we have $[b',a',a']$ or $[b',a',b']$ non-zero. So we may assume that $[b,a,a] \neq 0$, and that $[b,a,b]$, $[b,a,c]$, $[b,a,d]$ are scalar multiples of $[b,a,a]$.

Assume for the moment that

$$[b,a,b] = [b,a,c] = [b,a,d] = 0.$$

Then

$$\begin{aligned} [c,a,a] &= [c,a,b] = [c,a,c] = 0, \\ [c,a,d] &= -\omega[b,a,a]. \end{aligned}$$

Replacing b by $b+\lambda[b,a]$, c by $c+\mu[b,a]$ and d by $d+\nu[b,a]$ for suitable λ, μ, ν we may suppose that $[d,c] = \omega[b,a]$, $[d,b] = [c,a]$, $[d,a] = 0$. And if $[c,b]$ is non-zero, then by scaling we may assume that $[c,b] = [b,a,a]$. So we obtain two algebras [DW.135, DW.136]

$$\begin{aligned} \langle a,b,c,d \mid [d,a] &= [c,b] = 0,\ [c,a] = [d,b],\ [d,c] = \omega[b,a], \\ [b,a,b] &= [b,a,c] = [b,a,d] = 0,\ \text{class } 3\rangle, \end{aligned} \tag{7.53}$$

$$\begin{aligned} \langle a,b,c,d \mid [d,a] &= 0,\ [c,a] = [d,b],\ [d,c] = \omega[b,a],\ [c,b] = [b,a,a], \\ [b,a,b] &= [b,a,c] = [b,a,d] = 0,\ \text{class } 3\rangle. \end{aligned} \tag{7.54}$$

We show that 7.53 and 7.54 are the only possibilities by showing that we can always assume that

$$[b,a,b] = [b,a,c] = [b,a,d] = 0.$$

Since $[b,a,a] \neq 0$ we see that $[b,a]$ is centralized by $d + \lambda a$ for some λ. But

$$\begin{aligned} & [c,a,d+\lambda a] \\ =\ & [c,a,d] + \lambda[c,a,a] \\ =\ & -\omega[b,a,a] - \lambda[b,a,d] \\ =\ & (\lambda^2 - \omega)[b,a,a], \end{aligned}$$

and so (since ω is not a square) $[c,a]$ is not centralized by $d + \lambda a$. So the centralizers in L of $[b,a]$ and $[c,a]$ are distinct, and both have codimension 1. This implies that the centralizer, C, of L^2 has codimension 2. We show that $[C,C] \leq L^3$.

Suppose for the moment that $[C,C] \not\leq L^3$. We show that this implies that we can ønd new generators a,b,c,d for L satisfying $[d,a]$, $[c,b] \in L^3$ and $[d,b] = [c,a]$ modulo L^3, $[d,c] = \omega[b,a]$ modulo L^3, but with $a,c \in C$. However it is easy to see that if

$a, c \in C$ then $L^3 = \{0\}$, which gives a contradiction. We pick the new generators as follows. First we pick a and c linearly independent modulo L^2 so that $a, c \in C$. Every element outside the derived algebra of 6.7 has breadth 2, so we can ønd b, d so that a, b, c, d generate L and so that $[b, a]$, $[c, a]$ are linearly independant modulo L^3, and so that $[d, a] \in L^3$. Replacing b by $b + \lambda a$ and replacing c by $c + \mu a$ for suitable λ, μ we can assume that $[c, b] \in L^3$. Note that $c + \mu a \in C$. Following the same argument as in the construction of 6.7 we see that without further altering a or c (except by scaling) we can make further substitutions for b and d so that a, b, c, d satisfy the required conditions. This proves that $[C, C] \leq L^3$.

Since $C \geq L^2$ and C has codimension 2 in L, we can pick elements $b', c' \in C$ with b', c' linearly independant modulo L^2. Let

$$\begin{aligned} C_1 &= \{z \in L \mid [b', z] \in L^3\}, \\ C_2 &= \{z \in L \mid [c', z] \in L^3\}. \end{aligned}$$

Since $[C, C] \leq L^3$ it follows that $C_1, C_2 \geq C$. And since every element of 6.7 outside the derived algebra has breadth 2, it follows that C_1 and C_2 both have codimension 2 in L. So

$$C_1 = C_2 = C.$$

Furthermore if $\lambda \in \mathbb{Z}_p$ then

$$\{z \in L \mid [b' + \lambda c', z] \in L^3\} = C.$$

It follows that if a' is any element of L outside C then $[b', a']$, $[c', a']$ are linearly independent modulo L^3. Using the same arguments as were used in the construction of 6.7 we see that there are (new) generators a, b, c, d satisfying the relations above, with b, c linear combinations of b', c'. So $[d, a]$, $[c, b] \in L^3$ and $[d, b] = [c, a]$ modulo L^3, $[d, c] = \omega[b, a]$ modulo L^3, and

$$[b, a, b] = [b, a, c] = [c, a, b] = [c, a, c] = 0.$$

It is clear that d must centralize $\lambda[b, a] + \mu[c, a]$ for some λ, μ which are not both zero. But it is straightforward to check that algebra 6.7 has an automorphism mapping a to a, b to $\lambda b + \mu c$, c to $\omega\mu b + \lambda c$, d to d. So we may assume that

$$[b, a, b] = [b, a, c] = [b, a, d] = 0,$$

as claimed above.

Note that 7.53 has $p^3(p^2 - 1)$ elements of breadth 2, but that 7.54 has none. So these two algebras are diœerent.

5.4.7 case 7

Now suppose that L is an immediate descendant of 6.8. Then L is generated by a, b, c, d, L^2 is generated by $[b, a]$ modulo L^3, and L^3 is generated by $[b, a, b]$ modulo

L^4, and L^4 is generated by $[b,a,b,b]$. The commutators $[c,a]$, $[d,a]$, $[b,a,a]$, $[c,b]$, $[d,b]$, $[d,c]$ lie in L^4. Replacing c by $c+\lambda[b,a,b]$ for suitable λ we may suppose that $[c,b]=0$, and we may similarly suppose that $[d,b]=0$. We consider the cases $[d,c]=0$ and $[d,c]\neq 0$ separately.

First consider the case when $[d,c]=0$. We can choose d to lie in the centralizer of a, so that $[d,a]=0$, leaving $[c,a]$ and $[b,a,a]$ as possibly non-zero. By scaling we obtain four algebras [DW.8, DW.9, DW.22, DW.23]

$$\langle a,b\,|\,[b,a,a]=0,\ \text{class } 4\rangle \oplus \langle c\rangle \oplus \langle d\rangle, \tag{7.55}$$

$$\langle a,b\,|\,[b,a,a]=[b,a,b,b],\ \text{class } 4\rangle \oplus \langle c\rangle \oplus \langle d\rangle, \tag{7.56}$$

$$\langle a,b,c\,|\,[b,a,a]=0,\ [c,a]=[b,a,b,b],\ [c,b]=0,\ \text{class } 4\rangle \oplus \langle d\rangle, \tag{7.57}$$

$$\langle a,b,c\,|\,[b,a,a]=[b,a,b,b],\ [c,a]=[b,a,b,b],\ [c,b]=0,\ \text{class } 4\rangle \oplus \langle d\rangle. \tag{7.58}$$

Now suppose that $[d,c]\neq 0$. Scaling c we may suppose that $[d,c]=[b,a,b,b]$. Replacing a by $a+\alpha c+\beta b$ for suitable α,β we may suppose that $[c,a]=[d,a]=0$. And by scaling a and c we may suppose that $[b,a,a]=0$ or $[b,a,b,b]$. This gives [DW.112, DW.113]

$$\langle a,b\,|\,[b,a,a]=0,\ \text{class } 4\rangle \oplus_{[b,a,b,b]=[d,c]} \langle c,d\,|\,\text{class } 2\rangle, \tag{7.59}$$

$$\langle a,b\,|\,[b,a,a]=[b,a,b,b],\ \text{class } 4\rangle \oplus_{[b,a,b,b]=[d,c]} \langle c,d\,|\,\text{class } 2\rangle. \tag{7.60}$$

In 7.55, 7.57 and 7.59 the centralizer of the derived algebra has dimension 6, but in the other three algebras it has dimension 5. In 7.55 and 7.56 the centre has dimension 3, in 7.57 and 7.58 it has dimension 2, and in 7.59 and 7.60 it has dimension 1. So these 6 algebras are distinct.

5.4.8 case 8

Finally, suppose that L is an immediate descendant of 6.9. Then L is generated by a,b,c,d, L^2 is generated by $[b,a]$ modulo L^3, and L^3 is generated by $[b,a,b]$ modulo L^4, and L^4 is generated by $[b,a,b,b]$. The commutators $[d,a]$, $[b,a,a]$, $[c,b]$, $[d,b]$, $[d,c]$ lie in L^4 and $[c,a]=-[b,a,b]$ modulo L^4. The subalgebra generated by a,b,c is an immediate descendant of 5.6, and so is isomorphic to 6.26. So we can assume that $[b,a,a]=[c,b]=0$ and $[c,a]=-[b,a,b]$. Replacing d by $d+\lambda[b,a,b]$ for suitable λ we may assume that $[d,b]=0$. And by scaling a and d we may suppose that $[d,a]$ and $[d,c]$ are (independantly) equal to 0 or to $[b,a,b,b]$. So we have four algebras [DW.29, DW.118, DW.119, DW.120]

$$\langle a,b,c\,|\,[b,a,a]=[c,b]=0,\ [c,a]=-[b,a,b],\ \text{class } 4\rangle \oplus \langle d\rangle, \tag{7.61}$$

$$\begin{aligned}\langle a,b,c\,|\,[b,a,a] &= [c,b]=0,\ [c,a]=-[b,a,b],\\ [d,a] &= [d,b]=0,\ [d,c]=[b,a,b,b],\ \text{class } 4\rangle,\end{aligned} \tag{7.62}$$

$$\begin{aligned}\langle a,b,c \mid [b,a,a] &= [c,b]=0,\ [c,a]=-[b,a,b],\\ [d,b] &= [d,c]=0,\ [d,a]=[b,a,b,b],\ \text{class } 4\rangle,\end{aligned} \tag{7.63}$$

$$\begin{aligned}\langle a,b,c \mid [b,a,a] &= [c,b]=0,\ [c,a]=-[b,a,b],\\ [d,b] &= 0,\ [d,a]=[d,c]=[b,a,b,b],\ \text{class } 4\rangle.\end{aligned} \tag{7.64}$$

Algebra 7.61 is distinct from the other 3 since it has a centre of dimension 2, whereas in the other 3 algebras the centre has dimension 1. Algebra 7.62 has $(p-1)p(2p+1)$ elements of breadth 1, whereas 7.63 and 7.64 have $(p^2-1)p$. So it remains to show that 7.63 is distinct from 7.64. We note that in both these algebras

$$C = \langle c, d, [b,a], [b,a,b], [b,a,b,b]\rangle$$

is the inverse image in L of the centre of L/L^3, and that

$$D = \langle d, [b,a,b], [b,a,b,b]\rangle$$

is the inverse image in L of the centre of L/L^4. So both these subalgebras are characteristic. But in 7.63 $[C,D]=\{0\}$ and in 7.64 $[C,D]\neq\{0\}$.

5.5 3 generators

Let L be a three generator nilpotent Lie algebra of dimension 7. Then L is an immediate descendent of 5.4, or one of 6.11 ˘ 6.16 or 6.20 ˘ 6.26. (The algebras 5.5 and 5.6 do not have immediate descendants of dimension 7, and 6.17 ˘ 6.19 are terminal.)

5.5.1 case 1

Let L be an immediate descendant of 5.4. Then L is generated by a,b,c, L^2 is spanned modulo L^3 by $[b,a]$, $[c,a]$, and L^3 is spanned by $[b,a,a]$, $[b,a,b]$, $[b,a,c]=[c,a,b]$, $[c,a,a]$, $[c,a,c]$. We also have $[c,b]\in L^3$.

First we consider the case when some non-trivial linear combination of $[b,a]$ and $[c,a]$ is central. Clearly we may then assume that $[c,a]$ is central, so that L^3 is spanned by $[b,a,a]$ and $[b,a,b]$. Replacing c by $c+\lambda[b,a]$ for suitable λ we may suppose that $[c,b]=\alpha[b,a,a]$ for some α. By scaling c we may take $\alpha=0$ or 1. So we have two algebras [DW.41, DW.42]

$$\langle a,b,c \mid [c,b]=[c,a,a]=[c,a,b]=[c,a,c]=0,\ \text{class } 3\rangle, \tag{7.65}$$

$$\langle a,b,c \mid [c,a,a]=[c,a,b]=[c,a,c]=0,\ [c,b]=[b,a,a],\ \text{class } 3\rangle. \tag{7.66}$$

Next consider the possibility that

$$[b,a,b]=[b,a,c]=[c,a,b]=[c,a,c]=0.$$

Then L^3 is spanned by $[b,a,a]$ and $[c,a,a]$. If $[c,b]$ is non-zero then we may assume that $[c,b]=[b,a,a]$. So we have two algebras [DW.85, DW.86]

$$\langle a,b,c\,|\,[c,b]=[b,a,b]=[b,a,c]=[c,a,b]=[c,a,c]=0,\ \text{class } 3\rangle, \tag{7.67}$$

$$\langle a,b,c\,|\,[c,b]=[b,a,a],\ [b,a,b]=[b,a,c]=[c,a,b]=[c,a,c]=0,\ \text{class } 3\rangle. \tag{7.68}$$

Now consider the possibility that $[b,a,b]$, $[b,a,c]$, $[c,a,c]$ span a space of dimension 1. We may assume that $[b,a,b]\neq 0$ and that $[b,a,c]=0$ and that $[c,a,c]=\lambda[b,a,b]$ for some λ.

First consider the case when $\lambda=0$. Then we must have $[c,a,a]\neq 0$. If $[c,a,a]=\mu[b,a,b]$ then by scaling a we may assume that $\mu=1$. In this case $[b,a,a]$ and $[b,a,b]$ must be linearly independant, so that if $[c,b]=0$ then we have [DW.87]

$$\langle a,b,c\,|\,[c,b]=[b,a,c]=[c,a,c]=0,\ [c,a,a]=[b,a,b],\ \text{class } 3\rangle. \tag{7.69}$$

On the other hand, if $[c,b]\neq 0$, then replacing c by $c+\mu[b,a]$ for suitable μ we may suppose that $[c,b]=\nu[b,a,a]$ for some ν. By scaling we can assume that $\nu=1$. This gives [DW.88]

$$\langle a,b,c\,|\,[c,b]=[b,a,a],\ [b,a,c]=[c,a,c]=0,\ [c,a,a]=[b,a,b],\ \text{class } 3\rangle. \tag{7.70}$$

If $[c,a,a]$ and $[b,a,b]$ are linearly independant then $[b,a,a]=\alpha[b,a,b]+\beta[c,a,a]$ for some α,β. Repalcing a by $a'=a-\alpha b$ then we have $[b,a']=[b,a]$, $[c,a']=[c,a]$ modulo L^3 (with $[b,a',c]=[c,a',c]=0$). We also have $[c,a',a']=[c,a,a]$ and $[b,a',a']=\beta[c,a',a']$. Replacing b by $b'-\beta c$ we have $[b',a',a']=0$. As above, we may suppose that $[c,b]=\nu[c,a',a']$ for some ν. By scaling we can assume that $\nu=0$ or 1. So we have two algebras [DW.75, DW.78]

$$\langle a,b,c\,|\,[c,b]=[b,a,a]=[b,a,c]=[c,a,c]=0,\ \text{class } 3\rangle, \tag{7.71}$$

$$\langle a,b,c\,|\,[b,a,a]=[b,a,c]=[c,a,c]=0,\ [c,b]=[c,a,a],\ \text{class } 3\rangle. \tag{7.72}$$

Now let $[b,a,c]=0$ and let $[c,a,c]=\lambda[b,a,b]$ for some $\lambda\neq 0$. At least one of $[b,a,a]$ and $[c,a,a]$ must be linearly independant from $[b,a,b]$, and we can assume that $[b,a,a]$ and $[b,a,b]$ are linearly independant. So $[c,a,a]=\alpha[b,a,a]+\beta[b,a,b]$ for some α,β. Replacing a by $a-\mu c$ for suitable μ we may suppose that $\beta=0$, and scaling c we may assume that $\alpha=0$ or 1. (Note that the value of λ may change here, but that it is still non-zero.)

If $\alpha=0$ then we have

$$\begin{aligned}
[b,a,c] &= 0,\\
[c,a,c] &= \lambda[b,a,b]\neq 0,\\
[c,a,a] &= 0,\\
[c,b] &= \gamma[b,a,a]+\delta[b,a,b] \text{ for some } \gamma,\delta.
\end{aligned}$$

If we let $c' = c - \delta[b,a]$ then $[c',b] = \gamma[b,a,a]$, so we can assume that $\delta = 0$. Scaling c we may take $\gamma = 0$ or 1, and then scaling b we may assume that $\lambda = 1$ or ω. This gives [DW.76, DW.77, DW.79, DW.80]

$$\langle a,b,c \mid [c,b] = [b,a,c] = [c,a,a] = 0,\ [c,a,c] = [b,a,b],\ \text{class } 3\rangle, \tag{7.73}$$

$$\langle a,b,c \mid [c,b] = [b,a,c] = [c,a,a] = 0,\ [c,a,c] = \omega[b,a,b],\ \text{class } 3\rangle, \tag{7.74}$$

$$\langle a,b,c \mid [b,a,c] = [c,a,a] = 0,\ [c,b] = [b,a,a],\ [c,a,c] = [b,a,b],\ \text{class } 3\rangle, \tag{7.75}$$

$$\langle a,b,c \mid [b,a,c] = [c,a,a] = 0,\ [c,b] = [b,a,a],\ [c,a,c] = \omega[b,a,b],\ \text{class } 3\rangle. \tag{7.76}$$

On the other hand, if $\alpha = 1$ then provided $\lambda \neq -1$ we set $b' = \lambda b + c$, $c' = b - c$. Then $[b',a']$, $[c',a']$ span L^2 modulo L^3, and

$$\begin{aligned} [b',a,c'] &= 0, \\ [b',a,b'] &= \lambda(1+\lambda)[b,a,b], \\ [c',a,c'] &= (1+\lambda)[b,a,b], \\ [c',a,a] &= 0. \end{aligned}$$

So replacing b,c by b',c' we have

$$\begin{aligned} [b,a,c] &= 0, \\ [c,a,c] &= \lambda^{-1}[b,a,b] \neq 0, \\ [c,a,a] &= 0, \end{aligned}$$

and we are back in the case dealt with above.

If $\alpha = 1$ and $\lambda = -1$ then we have

$$\begin{aligned} [b,a,c] &= 0, \\ [c,a,c] &= -[b,a,b], \\ [c,a,a] &= [b,a,a], \\ [c,b] &= \gamma[b,a,a] + \delta[b,a,b] \end{aligned}$$

for some γ, δ. As above we can take $\delta = 0$, and scaling a,b,c by the same scale factor, we may assume that $\gamma = 0$ or 1. this gives [DW.81, DW.84]

$$\langle a,b,c \mid [c,b] = [b,a,c] = 0,\ [c,a,a] = [b,a,a],\ [c,a,c] = -[b,a,b],\ \text{class } 3\rangle, \tag{7.77}$$

$$\langle a,b,c \mid [b,a,c] = 0,\ [c,b] = [c,a,a] = [b,a,a],\ [c,a,c] = -[b,a,b],\ \text{class } 3\rangle. \tag{7.78}$$

Now assume that that $[b,a,b]$, $[b,a,c]$, $[c,a,c]$ span a space of dimension 2. Note that this implies that if d is a non-trivial linear combination of b and c then at least one of $[d,a,b]$, $[d,a,c]$ is non-zero. We consider two separate cases: the ørst when there is such an element d with $[d,a,b]$, $[d,a,c]$ linearly dependant, and the second when $[d,a,b]$ and $[d,a,c]$ are linearly independant for all such d.

In the ørst case we may suppose that $d=c$, so that $[c,a,b]$, $[c,a,c]$ are dependant. This implies that $[c,a,e]=0$ for some non-trivial linear combination e of b and c. If e and c are linearly independant then we can assume that $e=b$. So we have $[c,a,c]=0$ or $[c,a,b]=0$.

Suppose for the moment that $[c,a,b]=0$. Then $[b,a,c]=0$, and so L^3 is spanned by $[b,a,b]$ and $[c,a,c]$. Let

$$\begin{aligned}[b,a,a] &= \alpha[b,a,b]+\beta[c,a,c],\\ [c,a,a] &= \gamma[b,a,b]+\delta[c,a,c].\end{aligned}$$

Replacing a by $a-\alpha b-\delta c$ we have

$$\begin{aligned}[b,a,a] &= \beta[c,a,c],\\ [c,a,a] &= \gamma[b,a,b].\end{aligned}$$

One possibility is that $\beta=\gamma=0$. If one of β,γ is zero, then (swapping b and c if necessary) we may suppose that $\gamma=0$, and scaling b we may suppose that $\beta=1$. If β and γ are both non-zero, then scaling b we may suppose that $\beta=1$. Then setting $a'=\lambda a$, $b'=\lambda^{-1}\mu^2 b$, $c'=\mu c$ we have

$$\begin{aligned}[b',a',a'] &= \lambda\mu^2[b,a,a]=\lambda\mu^2[c,a,c]=[c',a',c'],\\ [c',a',a'] &= \lambda^2\mu[c,a,a]=\lambda^2\mu\gamma[b,a,b]=\lambda^3\mu^{-3}\gamma[b',a',b'].\end{aligned}$$

So if $p\neq 1 \bmod 3$ then we can scale a,b,c so that $\gamma=1$, but if $p=1 \bmod 3$ then we can scale a,b,c so that $\gamma=1$, ω or ω^2. However, $\gamma=\omega$ gives an isomorphic algebra to $\gamma=\omega^2$, for if

$$\begin{aligned}[b,a,a] &= [c,a,c],\\ [c,a,a] &= \omega^2[b,a,b],\end{aligned}$$

then setting $b'=c$, $c'=\omega b$ we have

$$\begin{aligned}[b',a,a] &= [c',a,c'],\\ [c',a,a] &= \omega[b',a,b'].\end{aligned}$$

In all these cases we can replace b by $b+\zeta[c,a]$ and replace c by $c+\eta[b,a]$ for suitable ζ,η, so that $[c,b]=0$. So we obtain the following algebras [DW.70, DW.73, DW.82, DW.83]

$$\langle a,b,c \mid [c,b]=[b,a,a]=[b,a,c]=[c,a,a]=0,\ \text{class } 3\rangle, \tag{7.79}$$

$$\langle a,b,c \mid [c,b]=[b,a,c]=[c,a,a]=0,\ [b,a,a]=[c,a,c],\ \text{class } 3\rangle, \tag{7.80}$$

$$\langle a,b,c \mid [c,b]=[b,a,c]=0,\ [b,a,a]=[c,a,c],\ [c,a,a]=[b,a,b],\ \text{class } 3\rangle, \tag{7.81}$$

and, in the case when $p=1 \bmod 3$,

$$\langle a,b,c \mid [c,b]=[b,a,c]=0,\ [b,a,a]=[c,a,c],\ [c,a,a]=\omega[b,a,b],\ \text{class } 3\rangle. \tag{7.82}$$

Next suppose that $[c,a,c]=0$ so that L^3 is spanned by by $[b,a,b]$, $[b,a,c]$. As above, we may suppose that $[c,b]=0$. The commutators $[b,a,a]$, $[c,a,a]$ are linear combination of $[b,a,b]$ and $[b,a,c]$, and we suppose that

$$\begin{aligned}[b,a,a] &= \alpha[b,a,b]+\beta[b,a,c],\\ [c,a,a] &= \gamma[b,a,b]+\delta[b,a,c].\end{aligned}$$

Replacing a by $a-\delta b-\beta c$ we have

$$\begin{aligned}[b,a,a] &= (\alpha-\delta)[b,a,b],\\ [c,a,a] &= \gamma[b,a,b].\end{aligned}$$

One possibility is that $\alpha-\delta=\gamma=0$. This gives [DW.89]

$$\langle a,b,c \mid [c,b]=[b,a,a]=[c,a,a]=[c,a,c]=0,\ \text{class } 3\rangle. \tag{7.83}$$

If $\alpha-\delta=0$ but $\gamma\neq 0$, then scaling c we may suppose that $\gamma=1$. Similarly, if $\alpha-\delta\neq 0$ but $\gamma=0$ we can scale a so that $[b,a,a]=[b,a,b]$. This gives two more algebras [DW.93, DW.91]

$$\langle a,b,c \mid [c,b]=[b,a,a]=[c,a,c]=0,\ [c,a,a]=[b,a,b],\ \text{class } 3\rangle, \tag{7.84}$$

$$\langle a,b,c \mid [c,b]=[c,a,a]=[c,a,c]=0,\ [b,a,a]=[b,a,b],\ \text{class } 3\rangle. \tag{7.85}$$

In the case when $\alpha-\delta$ and γ are both non-zero, scaling a and then c we may suppose that $[b,a,a]=[c,a,a]=[b,a,b]$, giving

$$\langle a,b,c \mid [c,b]=[c,a,c]=0,\ [b,a,a]=[b,a,b]=[c,a,a],\ \text{class } 3,\ p=3\rangle. \tag{7.85A}$$

However, if $p\neq 3$ then the following transformation shows that 7.85A is isomorphic to 7.84.

$$\begin{aligned}a' &= 9a-6b-2c,\\ b' &= 9b-3c,\\ c' &= 9c.\end{aligned}$$

Finally consider the case when $[d,a,b]$ and $[d,a,c]$ are linearly independant for all non-trivial linear combinations d of b,c. As above, we may suppose that $[c,b]=0$. We have

$$[c,a,c]=\alpha[b,a,b]+\beta[b,a,c] \text{ for some } \alpha,\beta.$$

If $\alpha=0$ then setting $d=c-\beta b$ we have $[d,a,c]=0$, contradicting out hypothesis. So $\alpha\neq 0$. Let $c'=c+\lambda b$. Then

$$[c',a,c']=(\lambda^2+\alpha)[b,a,b]+(2\lambda+\beta)[b,a,c].$$

So if we replace c by $c-\frac{\beta}{2}b$ we have $[c,a,c]=\gamma[b,a,b]$ for some γ. Our hypothesis implies that $\gamma\neq 0$, and scaling c we may suppose that $\gamma=1$ or ω. However, if $\gamma=1$ then $[b+c,a,b-c]=0$, contradicting our hypothesis. So we may assume that

$$[c,a,c]=\omega[b,a,b].$$

Replacing a by $a+\lambda b+\mu c$ for suitable λ,μ we may suppose that $[b,a,a]=0$. We then have

$$[c,a,a]=\nu[b,a,b]+\xi[b,a,c] \text{ for some } \nu,\xi.$$

One possibility is that $\nu=\xi=0$, giving [DW.90]

$$\langle a,b,c\,|\,[c,b]=[b,a,a]=[c,a,a]=0,\ [c,a,c]=\omega[b,a,b],\ \text{class } 3\rangle. \tag{7.86}$$

If $\nu=0$, $\xi\neq 0$ then scaling a we may take $\xi=1$, giving [DW.92]

$$\langle a,b,c\,|\,[c,b]=[b,a,a]=0,\ [c,a,a]=[b,a,c],\ [c,a,c]=\omega[b,a,b],\ \text{class } 3\rangle. \tag{7.87}$$

We now consider the general situation when $(\nu,\xi)\neq(0,0)$. We will show that when $p=1 \bmod 3$ then we have an algebra isomorphic to 7.87, but that when $p=2 \bmod 3$ then we have either 7.87 or one other algebra which we deøne below. We use the fact that if $p=1 \bmod 3$ then there is a solution to the equation $\alpha^2=-3$ in $\mathbb{Z}_p$, but if $p=2 \bmod 3$ then there is no solution to this equation. (This is because $a^2=-3$ if and only if $b^2+b+1=0$, where $b=(a-1)/2$.) First note that by scaling a, we can change (ν,ξ) to $(\lambda\nu,\lambda\xi)$ for any non-zero λ. So the isomorphism type of the algebra depends only on the ratio ν/ξ. If we let

$$\begin{aligned}
a' &= a+-\mu(\lambda^2-\omega\mu^2)^{-1}(\lambda\nu-\omega\mu\xi)b+\mu(\lambda^2-\omega\mu^2)^{-1}(\mu\nu-\lambda\xi)c,\\
b' &= \lambda b+\mu c,\\
c' &= \pm(\omega\mu b+\lambda c),
\end{aligned}$$

then

$$\begin{aligned}
[b',a',b'] &= (\lambda^2+\omega\mu^2)[b,a,b]+2\lambda\mu[b,a,c],\\
[c',a',c'] &= \omega[b',a',b'],\\
[b',a',c'] &= \pm(2\omega\lambda\mu[b,a,b]+(\lambda^2+\omega\mu^2)[b,a,c],\\
[b',a',a'] &= 0,\\
[c',a',a'] &= \pm\left((\lambda\nu-\omega\mu\xi)[b,a,b]-(\mu\nu-\lambda\xi)[b,a,c]\right)\\
&= \pm(\lambda^2-\omega\mu^2)^{-2}\left(\lambda v(\lambda^2+3\omega\mu^2)-\omega\mu\xi(3\lambda^2+\omega\mu^2)\right)[b',a',b']\\
&\quad -(\lambda^2-\omega\mu^2)^{-2}\left(\mu v(3\lambda^2+\omega\mu^2)-\lambda\xi(\lambda^2+3\omega\mu^2)\right)[b',a',c']
\end{aligned}$$

So we have another presentation of our algebra L in which the ratio ν/ξ is changed to

$$\pm\frac{\lambda v(\lambda^2+3\omega\mu^2)-\omega\mu\xi(3\lambda^2+\omega\mu^2)}{\mu v(3\lambda^2+\omega\mu^2)-\lambda\xi(\lambda^2+3\omega\mu^2)}.$$

Replacing μ by $-\mu$ this is

$$\pm\frac{\lambda v(\lambda^2+3\omega\mu^2)+\omega\mu\xi(3\lambda^2+\omega\mu^2)}{\mu v(3\lambda^2+\omega\mu^2)+\lambda\xi(\lambda^2+3\omega\mu^2)}. \qquad ((*))$$

If $p=3$ and we let $(\nu,\xi)=(1,0)$ then the ratio ν/ξ becomes

$$\pm\frac{\lambda^3}{\omega\mu^3}=\mp\frac{\lambda}{\mu}$$

which takes on all possible values. So we only get one group when $p=3$.

Next we consider the case when $p=1 \bmod 3$. If we let $(\nu,\xi)=(1,0)$ then the ratio ν/ξ becomes

$$\pm\frac{\lambda(\lambda^2+3\omega\mu^2)}{\mu(3\lambda^2+\omega\mu^2)}.$$

We show that when $p=1 \bmod 3$ this ratio takes all possible values in $\mathbb{Z}_p\cup\{\infty\}$. First note that the ration takes value 0 when $\lambda=0$ and it takes value ∞ when $\mu=0$. So we consider the case when λ,μ are both non-zero, and we let $\alpha=\lambda/\mu$, so that we can write

$$\frac{\lambda(\lambda^2+3\omega\mu^2)}{\mu(3\lambda^2+\omega\mu^2)}=\frac{\alpha(\alpha^2+3\omega)}{3\alpha^2+\omega}.$$

Note that since -3 is a square, this ratio is never 0 or ∞. We show that it takes on all possible values in $Z_p\backslash\{0\}$ by showing that

$$\frac{\alpha(\alpha^2+3\omega)}{3\alpha^2+\omega}=\frac{\beta(\beta^2+3\omega)}{3\beta^2+\omega} \qquad ((**))$$

only if $\alpha=\beta$. So suppose that equation (**) holds. Then

$$\begin{aligned} 0 &= \alpha(\alpha^2+3\omega)(3\beta^2+\omega)-\beta(\beta^2+3\omega)(3\alpha^2+\omega) \\ &= (\alpha-\beta)((\alpha-\beta)^2\omega+3(\alpha\beta-\omega)^2). \end{aligned}$$

Now $(\alpha-\beta)^2\omega+3(\alpha\beta-\omega)^2\neq 0$ if $\alpha\neq\beta$ since -3 is a square. So the ratio

$$\frac{\alpha(\alpha^2+3\omega)}{3\alpha^2+\omega}$$

takes on all values in $\mathbb{Z}_p$ as α ranges over $\mathbb{Z}_p$. This implies that if $(\nu,\xi)\neq(0,0)$ then L is isomorphic to 7.87.

Now we consider the case when $p=2 \bmod 3$. First we show that the ratio

$$\frac{\lambda(\lambda^2+3\omega\mu^2)}{\mu(3\lambda^2+\omega\mu^2)}$$

takes on $(p+1)/3$ values (including ∞). In the øeld of order p^2, $r\sqrt{\omega}+s$ is a cube if there are $\lambda, \mu \in \mathbb{Z}_p$ such that

$$r\sqrt{\omega}+s = (\lambda+\mu\sqrt{\omega})^3 = \lambda^3 + 3\lambda^2\mu\sqrt{\omega} + 3\lambda\omega\mu^2 + \mu^3\omega\sqrt{\omega}.$$

So

$$\begin{aligned} 3\lambda^2\mu + \mu^3\omega &= r, \\ \lambda^3 + 3\lambda\omega\mu^2 &= s, \end{aligned}$$

and

$$\frac{s}{r} = \frac{\lambda(\lambda^2+3\omega\mu^2)}{\mu(3\lambda^2+\omega\mu^2)}.$$

Since $p = 2 \bmod 3$, r is a cube in the øeld of p elements, so that $r\sqrt{\omega}+s$ is a cube if and only if $\sqrt{\omega}+s/r$ is a cube. So the values of

$$\frac{\lambda(\lambda^2+3\omega\mu^2)}{\mu(3\lambda^2+\omega\mu^2)}$$

are those $k \in \mathbb{Z}_p$ (together with ∞) such that $\sqrt{\omega}+k$ is a cube in the øeld of p^2 elements. There are $\frac{p+1}{3}$ of these values (including ∞). We consider the matrix

$$\begin{pmatrix} \lambda(\lambda^2+3\omega\mu^2) & \omega\mu(3\lambda^2+\omega\mu^2) \\ \mu(3\lambda^2+\omega\mu^2) & \lambda(\lambda^2+3\omega\mu^2) \end{pmatrix}$$

as an element of the group $PGL(2,p)$ acting on the ratios ν/ξ. Note that this matrix has determinant $(\lambda^2-\omega\mu^2)^3$, which is non-zero provided λ, μ are not both zero. It is straightforward to show set of these matrices forms a subgroup G of $PGL(2,p)$. Furthermore the eœect of the matrix above on the ratio ν/ξ depends on the value of

$$\frac{\lambda(\lambda^2+3\omega\mu^2)}{\mu(3\lambda^2+\omega\mu^2)}$$

(with the identity transformation corresponding to the value ∞). It is also easy to see that if $g \in G$ and $g \neq 1$, then g is øxed point free. So G has three orbits of size $(p+1)/3$ on $\mathbb{Z}_p \cup \{\infty\}$. Let Ω be the orbit of ∞. Then Ω contains all ratios

$$\frac{\lambda(\lambda^2+3\omega\mu^2)}{\mu(3\lambda^2+\omega\mu^2)},$$

and so $\Omega = -\Omega$. We show that if Φ, Ψ are the other two orbits then $\Phi = -\Psi$, so that equation (*) above implies that all the ratios in $\Phi \cup \Psi$ deøne isomorphic groups. To see this we need to show that if ν/ξ is in the same orbit as $-\nu/\xi$ then $\nu/\xi \in \Omega$. So suppose that

$$\frac{\lambda v(\lambda^2+3\omega\mu^2)+\omega\mu\xi(3\lambda^2+\omega\mu^2)}{\mu v(3\lambda^2+\omega\mu^2)+\lambda\xi(\lambda^2+3\omega\mu^2)} = -\frac{\nu}{\xi},$$

where $\nu/\xi \neq 0, \infty$. If we let
$$\alpha = \frac{\lambda(\lambda^2 + 3\omega\mu^2)}{\mu(3\lambda^2 + \omega\mu^2)},$$
and let $\beta = \nu/\xi$ then $\alpha \in \Omega\backslash\{\infty\}$ and $\beta \neq 0, \infty$, and we have
$$\frac{\alpha\beta + \omega}{\beta + \alpha} = -\beta.$$
So $\beta^2 + 2\alpha\beta + \omega = 0$, and
$$\frac{\beta^2 + \omega}{2\beta} = -\alpha \in \Omega.$$
This implies that
$$\sqrt{w} + \frac{\beta^2 + \omega}{2\beta}$$
is a cube in the øeld of p^2 elements, so that
$$2\beta\sqrt{\omega} + \beta^2 + \omega = (\sqrt{\omega} + \beta)^2 = \gamma^3$$
for some γ in the øeld of p^2 elements. But then
$$\sqrt{\omega} + \beta = \left(\frac{\sqrt{\omega} + \beta}{\gamma}\right)^3,$$
so that $\beta \in \Omega$ as claimed.

This shows that if $p = 2 \bmod 3$ then we have (at most) two isomorphism classes of algebras corresponding to values of (ν, ξ) other than $(0, 0)$ ¯ one with $\nu/\xi \in \Omega$ (giving 7.87), and one with $\nu/\xi \notin \Omega$. If we let k be any element of $\mathbb{Z}_p$ which is not a value of
$$\frac{\lambda(\lambda^2 + 3\omega\mu^2)}{\mu(3\lambda^2 + \omega\mu^2)},$$
then the other algebra is isomorphic to [DW.94]
$$\langle a, b, c \,|\, [c, b] = [b, a, a] = 0,\ [c, a, a] = [b, a, b]^k [b, a, c],\ [c, a, c] = \omega[b, a, b],\ \text{class } 3\rangle. \tag{7.88}$$

It remains to show that the algebras 7.65 ˘ 7.88 are all distinct. In most of these algebras we have $[c, b] = 0$, and so $\langle b, c\rangle$ is a two dimensional abelian subalgebra with trivial intersection with L^2. There is no such two dimensional subalgebra in the remaining cases: 7.66, 7.68, 7.70, 7.72, 7.75, 7.76, 7.78. So these algebras are distinct from the others. The algebras 7.65 and 7.66 have centres of dimension 3, but all the other algebras have centres of dimension 2. So 7.65 and 7.66 are distinct from the other algebras. They are also distinct from each other since $[c, b] = 0$ in 7.65, but not in 7.66. The dimension of the centralizer of L^2 in L in 7.67 ˘ 7.88 is given by the following table:

$\dim C_L(L^2)$	algebras
4	7.73 ˘ 7.78, 7.80 ˘ 7.82, 7.84, 7.85, 7.85A, 7.87, 7.88
5	7.69 ˘ 7.72, 7.79, 7.83, 7.86
6	7.67, 7.68

Note that this table, together with the criterion of whether $[c, b] = 0$ or not, distinguishes 7.67 and 7.68 from all other algebras (as well as from each other). The following table gives the number of elements of breadth 1 in the algebras 7.69 ˘ 7.86.

number of elements of breadth 1	algebras
0	7.81, 7.82, 7.84, 7.85A, 7.86 ˘ 7.88
$(p-1)p^2$	7.70, 7.73 ˘ 7.78, 7.80, 7.83, 7.85
$2(p-1)p^2$	7.72, 7.79
$(p^2-1)p^2$	7.69
$(p^2+p-2)p^2$	7.71

The criteria so far distinguish 7.65 ˘ 7.72 and 7.79 from each other, and from the other algebras. The subalgebra $C = \langle b, c\rangle + L^2$ is a characteristic subalgebra, and $[L^2, C]$ has dimension 1 in 7.69 ˘ 7.78 and dimension 2 in 7.79 ˘ 7.88. Also there exists an element $d \in C$ such that $[d, a, C]$ has dimension 1 in 7.79 ˘ 7.85A, but no such element in 7.86 ˘ 7.88. All the above criteria taken together distinguish 7.83 and 7.86 from all the other algebras, as well as from each other, and separate the remaining algebras which have not already been distinguished from the others into the following classes:

$$\{7.73, 7.74, 7.77\},\ \{7.75, 7.76, 7.78\},\ \{7.80, 7.85\},\ \{7.81, 7.82, 7.84, 7.85A\},\ \{7.87, 7.88\}.$$

Next note that the algebras 7.73 ˘ 7.76 have elements b, c which span C modulo L^2, and an element $a \notin C$ such that $[b, a, c] = [c, a, a] = 0$, whereas 7.77 and 7.78 have no such elements. This distinguishes 7.77 and 7.78. In 7.73 - 7.76 we let a' be any element of breadth 3 outside C (such as a), and quotient out the one dimensional central subalgebra $[L^2, a']$. The quotient algebra is 6.18 for 7.73 and 7.75, and 6.19 for 7.74 and 7.76. So the algebras 7.73 ˘ 7.78 are all distinct from each other.

Next, note that $[b, a, c] = 0$ in 7.80 ˘ 7.82 whereas in 7.84, 7.85, 7.85A and 7.86 it is impossible to ønd elements b, c which span C modulo L^2 and an element $a \notin C$ such that $[b, a, c] = 0$. This distinguishes each of the algebras 7.80 and 7.85 from all other algebras. But we still need to distinguish 7.84 from 7.85A.

We distinguisih between 7.81 and 7.82 when $p = 1 \bmod 3$ by noting that we have in eœect completely analysed all possible sets of generators a, b, c satsifying relations of the form

$$[c, b] = [b, a, c] = 0,\ [b, a, a] = [c, a, c],\ [c, a, a] = \gamma[b, a, b]$$

with $\gamma \neq 0$. (In addition to the arguments given above in this analysis you need to note that if b', c' span C modulo L^2, and if $[b', a', c'] = 0$ for some $a' \notin C$ then up to scale factors we either have $b' = b$ and $c' = c$ modulo L^2, or we have $b' = c$ and $c' = b$ modulo L^2.

And ønally, we note that if $p = 2$ mod 3 then 7.87 and 7.88 are distinct, since we have analysed all possible generating sets a, b, c satisfying relations of the form

$$[c, b] = [b, a, a] = 0,\ [c, a, a] = [b, a, b]\nu[b, a, c]\xi,\ [c, a, c] = \omega[b, a, b].$$

5.5.2 case 2

Next, let L be an immediate descendant of 6.11. Then L is generated by a, b, c, L^2 is spanned modulo L^3 by $[b, a]$, $[c, a]$, and $[c, b]$, and L^3 has dimension 1.

The centre of L can have dimension 1, 2 or 3. First we consider the case when the centre of L has dimension 3. We may suppose that $[c, a]$ and $[c, b]$ are central, so that L^3 is spanned by $[b, a, a]$ and $[b, a, b]$. Clearly we may assume that L^3 is spanned by $[b, a, a]$, and that $[b, a, b] = 0$. This gives [DW.40]

$$\langle a, b, c \,|\, [b, a, b] = [c, a, a] = [c, a, b] = [c, a, c] = [c, b, a] = [c, b, b] = [c, b, c] = 0,\ \text{class } 3\rangle. \tag{7.89}$$

Next, suppose that the centre of L has dimension 2. Then we may suppose that $[c, b]$ is central, so that L^3 is spanned by $[b, a, a]$, $[b, a, b]$, $[b, a, c] = [c, a, b]$, $[c, a, a]$, $[c, a, c]$. Note that we cannot have

$$[b, a, b] = [b, a, c] = [c, a, c] = 0$$

as this would imply that $[\lambda b + \mu c, a]$ was central for some λ, μ not both zero, so that the centre of L would have dimension 3. We show that in fact we can assume that one of $[b, a, b]$ and $[c, a, c]$ is non-zero. For suppose that $[b, a, b] = [c, a, c] = 0$ and $[b, a, c] = [c, a, b] \neq 0$. Then setting $b' = b + c$, $c' = b - c$ we have $[c', b']$ central, and $[b', a, c'] = 0$, so that at least one of $[b', a, b']$ and $[c', a, c']$ is non-zero. So suppose that $[b, a, b] \neq 0$, so that $[b, a, b]$ spans L^3. Replacing c by $c + \lambda b$ for suitable λ, we may suppose that $[b, a, c] = 0$. If $[c, a, c] \neq 0$ then scaling c we may suppose that $[c, a, c] = [b, a, b]$ or that $[c, a, c] = \omega[b, a, b]$. Replacing a by $a + \mu b + \nu c$ for suitable μ, ν we may suppose that $[b, a, a] = [c, a, a] = 0$. This gives [DW.71, DW.72]

$$\begin{aligned}\langle a, b, c \,|\, [b, a, a] &= [c, a, a] = [b, a, c] = [c, a, b] = [c, b, b] = [c, b, c] = 0, \\ [c, a, c] &= [b, a, b],\ \text{class } 3\rangle,\end{aligned} \tag{7.90}$$

$$\begin{aligned}\langle a, b, c \,|\, [b, a, a] &= [c, a, a] = [b, a, c] = [c, a, b] = [c, b, b] = [c, b, c] = 0, \\ [c, a, c] &= \omega[b, a, b],\ \text{class } 3\rangle.\end{aligned} \tag{7.91}$$

On the other hand, if $[c,a,c]=0$ then we must have $[c,a,a]\neq 0$, and by scaling c we may suppose that $[c,a,a]=[b,a,b]$. Replacing a by $a+\mu b$ for suitable μ we may suppose that $[b,a,a]=0$. So we have [DW.74]

$$\begin{aligned}\langle a,b,c\,|\,[b,a,a] &= [c,a,c]=[b,a,c]=[c,a,b]=[c,b,b]=[c,b,c]=0, \\ [c,a,a] &= [b,a,b], \text{ class } 3\rangle.\end{aligned} \quad (7.92)$$

To show that these three algebras are distinct we note that if L is 7.90 or 7.91 or 7.92 and $d\in\zeta(L)\backslash L^3$ then $L/\langle d\rangle$ is a 6 dimensional algebra isomorphic to 6.18 or 6.19 or 6.17 respectively.

Finally suppose that the centre of L has dimension 1, so that $\zeta(L)=L^3$. If $p=3$ we have the possibility that L is the free 2-Engel Lie algebra of rank 3, with presentation

$$\begin{aligned}\langle a,b,c\,|\,[b,a,a] &= [b,a,b]=[c,a,a]=[c,a,c]=[c,b,b]=[c,b,c]=0, \\ [a,b,c] &= [b,c,a]=[c,a,b], \text{ class } 3,\ p=3\rangle.\end{aligned} \quad (7.92\text{A})$$

If $p\neq 3$, we may assume that $[b,a,a]\neq 0$, and since L^3 has dimension 1 we may suppose that $[b,a,b]=[b,a,c]=0$. Suppose for the moment that $[c,a,b]\neq 0$. If $[c,a,a]=0$ we replace c by $b+c$ so that we now have $[b,a,b]=[b,a,c]=0$, with $[c,a,b]$ and $[c,a,a]$ both non-zero. Now replace b by $b+\lambda a$ for suitable λ and we have $[c,a,b]=0$, though we now have $[b,a,b]\neq 0$. So in every case we may assume that $[b,a,c]=[c,a,b]=0$ and that $[b,a,a]$ spans L^3. The commutators $[b,a,b]$, $[c,a,a]$, $[c,a,c]$, $[c,b,b]$, $[c,b,c]$ are all linear multiples of $[b,a,a]$.

Suppose for the moment that $[b,a,b]=0$. Then replacing c by $c+\mu b$ for suitable μ we may suppose that $[c,a,a]=0$. Since $[c,a]$ is not central we must have $[c,a,c]\neq 0$, and scaling b we may suppose that $[c,a,c]=[b,a,a]$. Since $[c,b]$ is not central, at least one of $[c,b,b]$, $[c,b,c]$ is non-zero. But if $[c,b,b]=0$ and $[c,b,c]=\nu[b,a,a]$ then $[c,b]-\nu[c,a]$ is central. So we may assume that $[c,b,b]=\nu[b,a,a]$ for some $\nu\neq 0$. Taking $a'=a$, $b'=\alpha^2 b$, $c'=\alpha c$ then we have

$$\begin{aligned}[b',a',b'] &= [b',a',c']=[c',a',a']=[c',a',b']=0, \\ [c',a',c'] &= \alpha^2[c,a,c]=\alpha^2[b,a,a]=[b',a',a'], \\ [c',b',b'] &= \alpha^5[c,b,b]=\alpha^5\nu[b,a,a]=\alpha^3\nu[b',a',a'].\end{aligned}$$

So if $p\neq 1 \bmod 3$ then we can choose α so that $[c',b',b']=[b',a',a']$ and if $p=1 \bmod 3$ then we can choose α so that $[c',b',b']=\xi[b',a',a']$ with $\xi=1$, ω or ω^2.

If $p\neq 1 \bmod 3$. Then we have [DW.126]

$$\begin{aligned}\langle a,b,c\,|\,[b,a,b] &= [b,a,c]=[c,a,a]=[c,a,b]=0,\ [c,a,c]=[b,a,a], \\ [c,b,b] &= [b,a,a],\ [c,b,c]=n[b,a,a], \text{ class } 3\rangle,\end{aligned} \quad (7.93)$$

(with $n\in\mathbb{Z}_p$). And if $p=1 \bmod 3$ then we have [DW.127, DW.128, DW.129]

$$\begin{aligned}\langle a,b,c\,|\,[b,a,b] &= [b,a,c]=[c,a,a]=[c,a,b]=0,\ [c,a,c]=[b,a,a], \\ [c,b,b] &= [b,a,a],\ [c,b,c]=n[b,a,a], \text{ class } 3\rangle,\end{aligned} \quad (7.94)$$

$$\begin{aligned} \langle a,b,c \mid [b,a,b] &= [b,a,c]=[c,a,a]=[c,a,b]=0,\ [c,a,c]=[b,a,a], \\ [c,b,b] &= \omega[b,a,a],\ [c,b,c]=n[b,a,a],\ \text{class } 3\rangle, \end{aligned} \tag{7.95}$$

$$\begin{aligned} \langle a,b,c \mid [b,a,b] &= [b,a,c]=[c,a,a]=[c,a,b]=0,\ [c,a,c]=[b,a,a], \\ [c,b,b] &= \omega^2[b,a,a],\ [c,b,c]=n[b,a,a],\ \text{class } 3\rangle, \end{aligned} \tag{7.96}$$

(with $n \in \mathbb{Z}_p$). In the case when $p = 1 \bmod 3$ then we can restrict the values of n as follows. If we pick $\alpha \in \mathbb{Z}_p$ with $\alpha^3 = 1$, and if we replace a, b, c by $a, \alpha^2 b, \alpha c$ in any of 7.94 ˘ 7.96, then all the relations except for the relation $[c, b, c] = n[b, a, a]$ stay the same, whereas this last relation becomes $[c, b, c] = n\alpha^{-1}[b, a, a]$. So we can restrict the non-zero values of n to $(p-1)/3$ values no two of which have the same cube.

Now suppose that $[b, a, c] = [c, a, b] = 0$. If any of the commutators $[b, a, a]$, $[b, a, b]$, $[c, a, a]$, $[c, a, c]$, $[c, b, b]$, $[c, b, c]$ are zero, then there is some permutation a', b', c' of the generators a, b, c such that $[b', a', b'] = 0$. Since $[b', a']$ is not central, this implies that $[b', a', a'] \neq 0$, and we are back in the case above. So we assume that all these commutators are non-zero. Scaling a we may suppose that $[b, a, a] = [b, a, b]$, and scaling c we have $[c, a, a] = [b, a, a]$. So

$$\begin{aligned} [b,a,c] &= [c,a,b]=0, \\ [b,a,a] &= [b,a,b]=[c,a,a], \\ [c,a,c] &= \lambda[b,a,a], \\ [c,b,b] &= \mu[b,a,a], \\ [c,b,c] &= \nu[b,a,a] \end{aligned}$$

for some non-zero λ, μ, ν. We show that we may assume that $\lambda = \mu = -1$, $\nu = 1$.

First we let $a' = a$, $b' = b + \alpha c$, $c' = b + \beta c$ for some $\alpha, \beta \in \mathbb{Z}_p$ with $\alpha \neq \beta$. Then

$$\begin{aligned} [b',a',c'] &= [c',a',b'] = (1+\alpha\beta\lambda)[b,a,a], \\ [b',a',a'] &= (1+\alpha)[b,a,a]. \end{aligned}$$

If we choose $\alpha = -1$, then provided $\lambda \neq -1$ we can take $\beta = \lambda^{-1}$, and then

$$[b',a',c'] = [c',a',b'] = [b',a',a'] = 0,$$

and we are back in the case above. So we assume that $\lambda = -1$. Similarly, we let $a' = a + \alpha c$, $b' = b$, $c' = a + \beta c$, and then

$$[b',a',c'] = [c',a',b'] = [c',a',a'] = 0$$

provided $1 - \alpha\beta\nu = 1 + \alpha\lambda = 0$. Now we have already shown that $\lambda = -1$, and so provided $\nu \neq 1$ we can take $\alpha = 1$, $\beta = \nu^{-1}$ and we are back in the case above. So we can assume that $\nu = 1$. Finally, we let $a' = a + \alpha b$, $b' = a + \beta b$, $c' = c$. Then

$$[b',a',c'] = [c',a',b'] = [b',a',a'] = 0$$

provided $1 + \alpha\beta\mu = 1 + \alpha = 0$. Provided $\mu \neq -1$ we can take $\alpha = -1$, $\beta = \mu^{-1}$ and we are back in the case above. So, as claimed above, we can assume that $\lambda = \mu = -1$, $\nu = 1$, and we have [DW.130]

$$\begin{aligned}\langle a, b, c \,|\, [b, a, c] &= [c, a, b] = 0,\ [b, a, a] = [b, a, b] = [c, a, a] = [c, b, c], \\ [c, a, c] &= [c, b, b] = -[b, a, a],\ \text{class } 3\rangle.\end{aligned} \quad (7.97)$$

It is clear that 7.97 is distinct from any of 7.93 ˘ 7.96, since 7.97 has $p + 2$ two dimensional subspaces of $Sp\langle a, b, c\rangle$ which generate a subalgebra of class 2 (and dimension 3), whereas 7.93 ˘ 7.96 have at most 3 such subspaces (the exact number depending on the value of n).

To show that there are no redundancies among the algebras 7.93 ˘ 7.96 we let L be a seven dimensional algebra of class 3, with generators a, b, c satisfying

$$\begin{aligned}[b, a, b] &= [b, a, c] = [c, a, a] = [c, a, b] = 0, \\ [c, a, c] &= [b, a, a], \\ [c, b, b] &= m[b, a, a], \\ [c, b, c] &= n[b, a, a],\end{aligned}$$

for some m, n with $m \neq 0$, and we consider all possible generating sets a', b', c' for L satisfying the relations

$$[b', a', b'] = [b', a', c'] = [c', a', a'] = [c', a', b'] = 0. \quad ((*))$$

We write

$$\begin{aligned}a' &= \alpha a + \beta b + \gamma c + u, \\ b' &= \varepsilon a + \zeta b + \eta c + v, \\ c' &= \lambda a + \mu b + \nu c + w,\end{aligned}$$

with $u, v, w \in L^2$. We note that the values of u, v, w make no diœerence to the relations satisøed by a', b', c', and so from now on we will take $u = v = w = 0$ for simplicity.

First we consider the situation when a', b', c' are scalar multiples of a, b, c (respectively). So

$$a' = \alpha a,\ b' = \zeta b,\ c' = \nu c.$$

Certainly the relations (*) are satisøed. To ensure that $[c', a', c'] = [b', a', a']$ we require $\alpha\nu^2 = \alpha^2\zeta$, so that $\zeta = \alpha^{-1}\nu^2$. Thus

$$\begin{aligned}[c', b', b'] &= \zeta^2\nu[c, b, b] = \zeta^2\nu m[b, a, a] = \alpha^{-3}\nu^3 m[b', a', a'], \\ [c', b', c'] &= \zeta\nu^2[c, b, c] = \zeta\nu^2 n[b, a, a] = \alpha^{-2}\nu^2 n[b, a, a].\end{aligned}$$

So scaling only enables us to alter the value of m by a cubic factor, and once the value of m is øxed it only enables us to alter the value of n by a factor which is a cube root of 1. So scaling does not lead to redundancies in 7.93 ˘ 7.96.

Next, we show that if a', b', c' generate L and satisfy (*), then (up to scalar multiples) b', c' are uniquely determined by a'. First note that since $[b', a', b'] = [b', a', c'] = 0$, $[b', a', a'] \neq 0$, since otherwise $[b', a']$ would be central. (We are using the fact that the centre of L is L^3.) Similarly $[c', a', c'] \neq 0$. Now suppose that a', b'', c'' generate L and satisfy

$$[b'', a', b''] = [b'', a', c''] = [c'', a', a'] = [c'', a', b''] = 0.$$

We want to show that b'', c'' are scalar multiples of b', c'. Let

$$\begin{aligned} b'' &= \varepsilon a' + \zeta b' + \eta c', \\ c'' &= \lambda a' + \mu b' + \nu c'. \end{aligned}$$

Since $[c'', a', a'] = 0$ we must have $\mu = 0$, and since c'' and a' must be linearly independant we must have $\nu \neq 0$. Then

$$0 = [c'', a', b'] = \eta\nu[c', a', c'],$$

and so $\eta = 0$. Since a' and b'' are linearly independant this implies that $\zeta \neq 0$. Then the fact that $[b'', a', b''] = 0$ implies that $\varepsilon = 0$, and the fact that $[b'', a', c''] = 0$ implies that $\lambda = 0$. So b'', c'' are scalar multiples of b', c', as claimed.

Finally note that if a', b', c' generate L and satisfy (*), then $[L, a', a'] \neq \{0\}$, since (as we showed above) $[b', a', a'] \neq 0$.

Now consider the general case when

$$a' = \alpha a + \beta b + \gamma c.$$

We let

$$\begin{aligned} b' &= m\gamma a + (\alpha + n\beta)b - m\beta c, \\ c' &= (m\beta + n\gamma)a - \gamma b + \alpha c. \end{aligned} \qquad ((**))$$

Then it is straightforward to check that a', b', c' satisfy exactly the same relations as a, b, c. However for certain values of a' it can happen that $[b', a', a'] = 0$, so that a', b', c' do not generate L. We show that if a', b' are as above, and if $[b', a', a'] = 0$ then it impossible to ønd b'', c'' such that a', b'', c'' generate L, and such that

$$[b'', a', b''] = [b'', a', c''] = [c'', a', a'] = [c'', a', b''] = 0.$$

So if a' is part of a generating set a', b', c' satisfying (*), then (up to scaling) b', c' must satisfy (**), and so the presentation for L in terms of a', b', c' is the same as the presentation for L in terms of a, b, c. (We are using the fact proved above that if

a', b', c' generate L and satisfy (*), then up to scaling b', c' are uniquely determined by a'.)

So suppose that $a' = \alpha a + \beta b + \gamma c$, that b', c' are given by (**), and that $[b', a', a'] = 0$. Note that, since a', b', c' satisfy the same relations as a, b, c, we also have $[b', a', b'] = 0$. Recall that if a' is part of a generating set for L satisfying (*) then there is no element $d \in Sp\langle a, b, c\rangle$ such that a, d are linearly independant and such that $[d, a, a] = [d, a, d] = 0$. So either a' is not part of a generating set for L satisfying (*) (which is what we want to prove), or a' and b' are linearly dependant. So assume that a' and b' are linearly dependant. Then

$$\begin{aligned} \alpha^2 + n\alpha\beta &= m\beta\gamma, \\ -m\alpha\beta &= m\gamma^2, \\ -m\beta^2 &= \alpha\gamma + n\beta\gamma. \end{aligned}$$

Using the fact that $m \neq 0$, it is easy to see that these equations imply that

$$[a, a', a'] = [b, a', a'] = [c, a', a'] = 0,$$

and, as we showed above, this implies that a' is not part of a generating set for L satisfying (*).

So there is no redundancy among the algebras 7.93 ˘ 7.96.

5.5.3 case 3

Now, let L be an immediate descendant of 6.12. Then L is generated by a, b, c, and the subalgebra generated by a, b has dimension 6, and is an immediate descendant of 5.7. So the subalgebra A generated by a, b is 6.27, 6.28 or 6.29. If A is 6.27 then $[b, a, a, a] = [b, a, a, b] = 0$, and L^4 is spanned by $[b, a, b, b]$. The commutators $[c, a]$, $[c, b]$ are linear multiples of $[b, a, b, b]$, and replacing c by $c + \lambda[b, a, b]$ for suitable λ we may suppose that $[c, b] = 0$. By scaling c we may suppose that $[c, a] = 0$ or $[b, a, b, b]$. So we have two algebras [DW.20, DW.59]

$$\langle a, b \mid [b, a, a, a] = [b, a, a, b] = 0, \text{ class } 4\rangle \oplus \langle c\rangle, \tag{7.98}$$

$$\langle a, b, c \mid [b, a, a, a] = [b, a, a, b] = 0, [c, a] = [b, a, b, b], [c, b] = 0, \text{ class } 4\rangle. \tag{7.99}$$

If A is 6.28 or 6.29 then L^4 has dimension 1 and is spanned by either of $[b, a, a, a]$, $[b, a, b, b]$, with $[b, a, a, b] = 0$. So we may suppose that $[c, a] = [c, b] = 0$ and we have [DW.27, DW.28]

$$\langle a, b \mid [b, a, a, b] = 0, [b, a, b, b] = [b, a, a, a], \text{ class } 4\rangle \oplus \langle c\rangle, \tag{7.100}$$

$$\langle a, b \mid [b, a, a, b] = 0, [b, a, b, b] = \omega[b, a, a, a], \text{ class } 4\rangle \oplus \langle c\rangle. \tag{7.101}$$

These four algebras are clearly distinct, since in three of them c is central, and in these three cases factoring out $\langle c\rangle$ gives diœerent algebras.

5.5.4 case 4

Next, let L be an immediate descendant of 6.13. Then L is generated by a, b, c, and (as in the case above) the subalgebra A generated by a, b has dimension 6, and is an immediate descendant of 5.7. So A is isomorphic to 6.27, 6.28 or 6.29. We have $[c, a] = -[b, a, b]$ modulo L^4 and $[c, b] \in L^4$.

First suppose that A is isomorphic to 6.27. Then there is some non-trivial linear combination of $[b, a, a]$ and $[b, a, b]$ which is central in A. We may suppose that this central element is $[b, a, a]$ or $[b, a, b]$. If $[b, a, a]$ is central then L^4 is spanned by $[b, a, b, b]$ and we have

$$\begin{aligned} [c, a] &= -[b, a, b] + \lambda[b, a, b, b], \\ [c, b] &= \mu[b, a, b, b] \end{aligned}$$

for some λ, μ.

Replacing c by $c - \mu[b, a, b]$ we may suppose that $[c, b] = 0$. Then, if we let $b' = b + \nu c$, we have

$$\begin{aligned} [b', a] &= [b, a] - \nu[b, a, b] + \lambda\nu[b, a, b, b], \\ [b', a, a] &= [b, a, a], \\ [b', a, b'] &= [b, a, b] - 2\nu[b, a, b, b], \\ [b', a, b', b'] &= [b, a, b, b], \\ [c, b'] &= 0. \end{aligned}$$

So if we replace b by $b - (\lambda/2)c$ then $[c, a] = -[b, a, b]$. So we have [DW.99]

$$\langle a, b, c \,|\, [b, a, a, a] = [b, a, a, b] = 0,\ [c, a] = -[b, a, b],\ [c, b] = 0,\ \text{class } 4\rangle. \tag{7.102}$$

On the other hand, if $[b, a, b]$ is central then L^4 is spanned by $[b, a, a, a]$ and we have

$$\begin{aligned} [c, a] &= -[b, a, b] + \lambda[b, a, a, a], \\ [c, b] &= \mu[b, a, a, a] \end{aligned}$$

for some λ, μ. Replacing c by $c - \lambda[b, a, a]$ we have $[c, a] = -[b, a, b]$, and by scaling we can take $\mu = 0$ or 1. So we have [DW.66, DW.69]

$$\langle a, b, c \,|\, [b, a, a, b] = [b, a, b, b] = 0,\ [c, a] = -[b, a, b],\ [c, b] = 0,\ \text{class } 4\rangle, \tag{7.103}$$

$$\langle a, b, c \,|\, [b, a, a, b] = [b, a, b, b] = 0,\ [c, a] = -[b, a, b],\ [c, b] = [b, a, a, a],\ \text{class } 4\rangle. \tag{7.104}$$

Next, suppose that A is isomorphic to 6.28 or 6.29 so that no non-trivial linear combination of $[b, a, a]$ and $[b, a, b]$ is central. If $[b, a, b, b] \neq 0$ then we can choose a so that $[b, a, b, a] = [b, a, a, b] = 0$. Then by scaling we have

$$[b, a, b, b] = \alpha[b, a, a, a] \text{ with } \alpha = 1 \text{ or } \omega.$$

Replacing c by $c+\lambda[b,a,a]+\mu[b,a,b]$ for suitable λ,μ we can assume that $[c,a]=-[b,a,b]$, $[c,b]=0$. This gives [DW.137, DW.138]

$$\langle a,b,c\,|\,[b,a,a,b]=0,\ [c,a]=-[b,a,b],\ [c,b]=0,\ [b,a,b,b]=[b,a,a,a],\ \text{class } 4\rangle, \tag{7.105}$$

$$\langle a,b,c\,|\,[b,a,a,b]=0,\ [c,a]=-[b,a,b],\ [c,b]=0,\ [b,a,b,b]=\omega[b,a,a,a],\ \text{class } 4\rangle. \tag{7.106}$$

But if $[b,a,b,b]=0$ then $[b,a,a,b]=[b,a,b,a]$ must be non-zero. Letting $a'=a+\beta b$ we have

$$\begin{aligned}[c,a'] &= -[b,a',b] \text{ modulo } L^4,\\ [b,a',a',a'] &= [b,a,a,a]+2\beta[b,a,a,b].\end{aligned}$$

We can choose β so that $[b,a',a',a']=0$, and then replacing c by $c+\lambda[b,a,a]+\mu[b,a,b]$ for suitable λ,μ we can assume that $[c,a']=-[b,a',b]$, $[c,b]=0$. This gives [DW.139]

$$\langle a,b,c\,|\,[b,a,a,a]=[b,a,b,b]=0,\ [c,a]=-[b,a,b],\ [c,b]=0,\ \text{class } 4\rangle. \tag{7.107}$$

We need to show that the algebras 7.102 ˘ 7.107 are distinct. ørst note that in 7.102 ˘ 7.104 the centre has dimension 2, whereas it has dimension 1 in 7.105 ˘ 7.107. Next note that $C=\langle c\rangle+L^2$ is a characteristic subalgebra, and that $[C,L^2]$ has dimension 0 in 7.103, 7.104 and 7.107, but dimension 1 in the other three algebras. So it only remains to show that 7.103 is not isomorphic to 7.104, and that 7.105 is not isomorphic to 7.106. To see that 7.103 is not isomorphic to 7.104 note that c has breadth 1 in 7.103, but that no element of $C\backslash L^2$ has breadth 1 in 7.104. And to see that 7.105 is distinct from 7.106 note that it is possible to ønd elements a',b' such that

$$\begin{aligned}[b',a',a',a'] &= [b',a',b',b']=0,\\ [b',a',a',b'] &\neq 0\end{aligned}$$

in one of the two algebras, but not in the other. (Which algebra has such elements depends on the value of $p \bmod 4$.)

5.5.5 case 5

Now let L be an immediate descendant of 6.14. Then L is generated by a,b,c, and (just as in the case the last two cases) the subalgebra A generated by a,b has dimension 6, and is an immediate descendant of 5.7. So A is isomorphic to 6.27, 6.28 or 6.29. We have $[c,a]=-[b,a,a]$ modulo L^4 and $[c,b]\in L^4$.

First suppose that A is isomorphic to 6.27. Then there is some non-trivial linear combination of $[b,a,a]$ and $[b,a,b]$ which is central in A. We may suppose that this central element is $[b,a,a]$, $[b,a,b]$ or $[b,a,a]-[b,a,b]$.

If $[b,a,a]$ is central in A then L^4 is spanned by $[b,a,b,b]$ and we have

$$\begin{aligned}[c,a] &= -[b,a,a]+\lambda[b,a,b,b],\\ [c,b] &= \mu[b,a,b,b]\end{aligned}$$

for some λ,μ. Replacing c by $c-\mu[b,a,b]$ we can assume that $[c,b]=0$. And scaling a and c we can suppose that $\lambda=0$ or 1. This gives [DW.62, DW.63]

$$\langle a,b,c\,|\,[b,a,a,a]=[b,a,a,b]=0,\ [c,a]=-[b,a,a],\ [c,b]=0,\ \text{class } 4\rangle, \tag{7.108}$$

$$\langle a,b,c\,|\,[b,a,a,a]=[b,a,a,b]=0,\ [c,a]=-[b,a,a]+[b,a,b,b],\ [c,b]=0,\ \text{class } 4\rangle. \tag{7.109}$$

On the other hand, if $[b,a,b]$ is central in A then L^4 is spanned by $[b,a,a,a]$ and we have

$$\begin{aligned}[c,a] &= -[b,a,a]+\lambda[b,a,a,a],\\ [c,b] &= \mu[b,a,a,a].\end{aligned}$$

We let $a'=b,\ b'=-a,\ c'=-c-[b,a]$. Then

$$\begin{aligned}[b',a'] &= [b,a],\\ [b',a',a'] &= [b,a,b],\\ [b',a',b'] &= -[b,a,a],\\ [c',a'] &= -[b',a',a'] \text{ modulo } L^4,\\ [c',b'] &\in L^4,\end{aligned}$$

and $[b',a',a']$ is central, so that we are back in the case dealt with above.

And if $[b,a,a]-[b,a,b]$ is central in A then

$$[b,a,a,a]=[b,a,a,b]=[b,a,b,b],$$

and

$$\begin{aligned}[c,a] &= -[b,a,a]+\lambda[b,a,a,a],\\ [c,b] &= \mu[b,a,a,a]\end{aligned}$$

for some λ,μ. Replacing b by $b-\lambda[b,a]$ we may suppose that

$$\begin{aligned}[c,a] &= -[b,a,a],\\ [c,b] &= \nu[b,a,a,a]\end{aligned}$$

for some ν. Now let $a'=a,\ b'=b-\nu c,\ c'=c-\nu[b,a,b]$. Then

$$\begin{aligned}[b',a'] &= [b,a]+\nu[b,a,a],\\ [b',a',a'] &= [b,a,a]+\nu[b,a,a,a],\\ [b',a',b'] &= [b,a,b]+2\nu[b,a,a,a],\\ [b',a',a',a'] &= [b',a',a',b']=[b',a',b',b']=[b,a,a,a],\\ [c',a'] &= -[b,a,a]-\nu[b,a,a,a]=-[b',a',a'],\\ [c',b'] &= 0.\end{aligned}$$

So replacing a, b, c by a', b', c' we have [DW.100]

$$\langle a, b, c \mid [b,a,a,a] = [b,a,a,b] = [b,a,b,b],\ [c,a] = -[b,a,a],\ [c,b] = 0,\ \text{class } 4\rangle. \tag{7.110}$$

Next, suppose that A is isomorphic to 6.28 or 6.29 so that no non-trivial linear combination of $[b,a,a]$ and $[b,a,b]$ is central. If $[b,a,a,a] \neq 0$ and $[b,a,a,b] = [b,a,b,a] = 0$, then by scaling we have

$$[b,a,b,b] = \alpha[b,a,a,a] \text{ with } \alpha = 1 \text{ or } \omega.$$

Replacing c by $c + \lambda[b,a,a] + \mu[b,a,b]$ for suitable λ, μ we can assume that $[c,a] = -[b,a,a]$, $[c,b] = 0$. This gives [DW.144, DW.145]

$$\langle a, b, c \mid [b,a,a,b] = 0,\ [c,a] = -[b,a,a],\ [c,b] = 0,\ [b,a,b,b] = [b,a,a,a],\ \text{class } 4\rangle, \tag{7.111}$$

$$\langle a, b, c \mid [b,a,a,b] = 0,\ [c,a] = -[b,a,a],\ [c,b] = 0,\ [b,a,b,b] = \omega[b,a,a,a],\ \text{class } 4\rangle. \tag{7.112}$$

But if $[b,a,a,a] = 0$ and $[b,a,a,b] = [b,a,b,a]$ are non-zero, then $[b,a,b,b] = \nu[b,a,a,b]$ where by scaling we may suppose that $\nu = 0$ or 1. Replacing c by $c+\lambda[b,a,a]+\mu[b,a,b]$ for suitable λ, μ we can assume that $[c,a] = -[b,a,a]$, $[c,b] = 0$. This gives [DW.146, DW.147]

$$\langle a, b, c \mid [b,a,a,a] = [b,a,b,b] = 0,\ [c,a] = -[b,a,a],\ [c,b] = 0,\ \text{class } 4\rangle, \tag{7.113}$$

$$\langle a, b, c \mid [b,a,a,a] = 0,\ [c,a] = -[b,a,a],\ [c,b] = 0,\ [b,a,b,b] = [b,a,a,b],\ \text{class } 4\rangle. \tag{7.114}$$

Finally, if $[b,a,a,a]$ and $[b,a,a,b]$ are both non-zero, then by scaling we may suppose that $[b,a,a,a] = [b,a,a,b]$, and that $[b,a,b,b] = \alpha[b,a,a,a]$ for some α. We also have

$$\begin{aligned} [c,a] &= -[b,a,a] + \lambda[b,a,a,a], \\ [c,b] &= \mu[b,a,a,a] \end{aligned}$$

for some λ, μ. Replacing b by $b - \lambda[b,a]$ we may suppose that

$$\begin{aligned} [c,a] &= -[b,a,a], \\ [c,b] &= \nu[b,a,a,a] \end{aligned}$$

for some ν. Now let $a' = a$, $b' = b - \nu c$, $c' = c - \nu[b,a,a]$. Then

$$\begin{aligned} [b',a'] &= [b,a] + \nu[b,a,a], \\ [b',a',a'] &= [b,a,a] + \nu[b,a,a,a], \\ [b',a',b'] &= [b,a,b] + 2\nu[b,a,a,a], \\ [b',a',a',a'] &= [b',a',a',b'] = [b,a,a,a], \\ [b',a',b',b'] &= \alpha[b,a,a,a] = \alpha[b',a',a',a'], \\ [c',a'] &= -[b,a,a] - \nu[b,a,a,a] = -[b',a',a'], \\ [c',b'] &= 0. \end{aligned}$$

So replacing a, b, c by a', b', c' we have

$$\begin{aligned}\langle a, b, c \,|\, [b, a, a, a] &= [b, a, a, b],\ [b, a, b, b] = \alpha[b, a, a, a], \\ [c, a] &= -[b, a, a],\ [c, b] = 0,\ \text{class } 4\rangle\ (\alpha \in \mathbb{Z}_p).\end{aligned} \tag{7.115}$$

Note that if $\alpha = 1$ this is 7.110 [DW.100]. If $\alpha = 0$ it is 7.114 [DW.147], and for $2 \leq \alpha < p$ it is DW.148.

We now show that the algebras 7.108 ˘ 7.115 are distinct (except that 7.115 with $\alpha = 1$ is 7.110, and with $\alpha = 0$ it is 7.114). First note that the centre of L has dimension 2 in 7.108 ˘ 7.110, and dimension 1 in 7.111 ˘ 7.115 (except when $\alpha = 1$ in 7.115). Next, note that $C = \langle c\rangle + L^2$ is the inverse image in L of the centre of L/L^3, so that it is a characteristic subalgebra. The subalgebra $[C, L^2]$ is spanned by $[b, a, a, b]$, and so has dimension 1 in 7.110, 7.113 ˘ 7.115, but dimension 0 in the remaining algebras. The subalgebra A generated by a, b is isomorphic to 6.27, 6.28 or 6.29, and if B is any subalgebra generated by two elements a', b' which are linearly independant modulo C, then $B \cong A$. In 7.111 the subalgebra A is isomorphic to 6.28, and in 7.112 it is isomorphic to 6.29, so 7.111 and 7.112 are not isomorphic. From the observations made so far, we see that 7.110, 7.111 and 7.112 are distinct from the other algebras, and from each other. Furthermore, the algebras 7.108 ˘ 7.109 are distinct from 7.113 ˘ 7.115. Algebra 7.108 has $(p-1)p^2(p+2)$ elements of breadth 1, and 7.109 has $2(p-1)p^2$ elements of breadth 1. So these two algebras are distinct from each other.

It remains to distinguish 7.113, 7.114 and 7.115 ($2 \leq \alpha < p$) from each other. We note that $c + L^4$ has breadth 1 in L/L^4, and we also note that the only elements in C/L^4 outside L^2/L^4 which have breadth 1 are of the form $\alpha c + u + L^4$ or of the form $\alpha(c + [b, a]) + u + L^4$ ($\alpha \neq 0$, $u \in L^3$). So, up to scaling, the only generators of L which satisfy the relations of L modulo L^4 are a', b', c' where

$$\begin{aligned} a' &= a \bmod C, \\ b' &= b \bmod C, \\ c' &= c \bmod L^3, \end{aligned}$$

or where

$$\begin{aligned} a' &= b \bmod C, \\ b' &= a \bmod C, \\ c' &= c + [b, a] \bmod L^3. \end{aligned}$$

In both cases, the commutators $[b', a', a', a']$, $[b', a', a', b']$, $[b', a', b', b']$ are non-zero scale multiples of the commutators $[b, a, a, a]$, $[b, a, a, b]$, $[b, a, b, b]$ (though not necessarily in the same order). In 7.113 two of these commutators are zero, in 7.114 (and in 7.115 with $\alpha = 0$) one of these commutators is zero, and in 7.115 with $\alpha \neq 0$ none of the commutators is zero. So 7.113 and 7.114 are distinct from each other, and also distinct from 7.115 for $2 \leq \alpha < p$.

Finally, consider the algebra 7.115 ($2 \leq \alpha < p$). As described above, if a', b', c' are generators of L which staify the relation of L modulo L^4 then either $a' = \lambda a \bmod C$, $b' = \mu b \bmod C$, or $a' = \mu b \bmod C$, $b' = \lambda a \bmod C$ (for some non-zero λ, μ). We consider the relations $[b, a, a, a] = [b, a, a, b]$ and $[b, a, b, b] = \alpha[b, a, a, a]$ of L. We would like to investigate the possibility that $[b', a', a', a'] = [b', a', a', b']$ but that $[b', a', b', b'] = \alpha'[b', a', a', a']$ with $\alpha' \neq \alpha$.

Suppose $a' = \lambda a \bmod C$, $b' = \mu b \bmod C$. If the relation $[b', a', a', a'] = [b', a', a', b']$ is satisøed then we must have $\lambda = \mu$. But then $[b', a', b', b'] = \alpha[b', a', a', a']$. And if $a' = \mu b \bmod C$, $b' = \lambda a \bmod C$ then

$$\begin{aligned}
[b', a', a', a'] &= -\lambda\mu^3[b, a, b, b] = -\lambda\mu^3\alpha[b, a, a, a], \\
[b', a', a', b'] &= -\lambda^2\mu^2[b, a, a, b] = -\lambda^2\mu^2[b, a, a, a], \\
[b', a', b', b'] &= -\lambda^3\mu[b, a, a, a].
\end{aligned}$$

To ensure that $[b', a', a', a'] = [b', a', a', b']$ we need $\lambda = \mu\alpha$, so that

$$[b', a', b', b'] = -\mu^4\alpha^3[b, a, a, a] = \alpha[b', a', a', a'].$$

So the algebras 7.115 with $2 \leq \alpha < p$ are all distinct.

5.5.6 case 6

Now let L be an immediate descendant of 6.15. Then L is generated by a, b, c, and (just as in the case the last three cases) the subalgebra A generated by a, b has dimension 6, and is an immediate descendant of 5.7. So A is isomorphic to 6.27, 6.28 or 6.29. We have $[c, a] = -[b, a, b]$ modulo L^4 and $[c, b] = -\omega[b, a, a]$ modulo L^4.

First suppose that A is isomorphic to 6.27, so that some non-trivial linear combination of $[b, a, a]$ and $[b, a, b]$ is central. There is no loss in generality in assuming that $[b, a, b]$ is central, and so we may assume that

$$\begin{aligned}
[b, a, a, b] &= [b, a, b, b] = 0, \\
[c, a] &= -[b, a, b] + \lambda[b, a, a, a], \\
[c, b] &= -\omega[b, a, a] + \mu[b, a, a, a]
\end{aligned}$$

for some λ, μ. Replacing c by $c - \lambda[b, a, a]$ we have

$$\begin{aligned}
[b, a, a, b] &= [b, a, b, b] = 0, \\
[c, a] &= -[b, a, b], \\
[c, b] &= -\omega[b, a, a] + \mu[b, a, a, a].
\end{aligned}$$

Now setting $b' = b + \alpha[b, a]$ we have

$$\begin{aligned}
[b', a] &= [b, a] + \alpha[b, a, a], \\
[b', a, a] &= [b, a, a] + \alpha[b, a, a, a], \\
[b', a, b'] &= [b, a, b], \\
[b', a, a, a] &= [b, a, a, a], \\
[c, b'] &= -\omega[b, a, a] + \mu[b, a, a, a] - \omega\alpha[b, a, a, a].
\end{aligned}$$

So if we set $\alpha = \mu(1 + \omega)^{-1}$ we have $[c, b'] = [b', a, a]$, and replacing b by b' we have [DW.102]

$$\langle a, b, c \,|\, [b, a, a, b] = [b, a, b, b] = 0,\ [c, a] = -[b, a, b],\ [c, b] = -\omega[b, a, a],\ \text{class } 4\rangle. \tag{7.106}$$

Next suppose that A is isomorphic to 6.28 or 6.29, so that no non-trivial linear combination of $[b, a, a]$ and $[b, a, b]$ is central. So there are generators a', b' for A with $[b', a', a', b'] = 0$, and $[b', a', a', a']$, $[b', a', b', b']$ non-zero. There is no loss in generality in assume that $a' = a$, so that $[b, a, a, a]$ spans L^4. So

$$\begin{aligned}
[b, a, a, b] &= \lambda[b, a, a, a], \\
[b, a, b, b] &= \mu[b, a, a, a]
\end{aligned}$$

for some λ, μ. To ensure that no non-trivial linear combination of $[b, a, a]$ and $[b, a, b]$ is central we require that $\mu \neq \lambda^2$. We also have

$$\begin{aligned}
[c, a] &= -[b, a, b] + \alpha[b, a, a, a], \\
[c, b] &= -\omega[b, a, a] + \beta[b, a, a, a]
\end{aligned}$$

for some α, β. If we let $c' = c + \gamma[b, a, a] + \delta[b, a, b]$ then

$$\begin{aligned}
[c', a] &= -[b, a, b] + (\alpha + \gamma + \lambda\delta)[b, a, a, a], \\
[c', b] &= -\omega[b, a, a] + (\beta + \lambda\gamma + \mu\delta)[b, a, a, a].
\end{aligned}$$

Since $\mu \neq \lambda^2$ we can find γ, δ so that

$$\alpha + \gamma + \lambda\delta = \beta + \lambda\gamma + \mu\delta = 0.$$

So, replacing c by c', we may suppose that

$$\begin{aligned}
[c, a] &= -[b, a, b], \\
[c, b] &= -\omega[b, a, a].
\end{aligned}$$

We consider all possible generating sets a', b', c' for L which satisfy the relation of L in a, b, c modulo L^4, and we analyze the orbits of the pair of parameters λ, μ under these different generating sets. If we let $C = \langle c\rangle + L^2$, then

$$\begin{aligned}
a' &= \alpha a + \beta b \bmod C, \\
b' &= \pm(\omega\beta a + \alpha b) \bmod C, \\
c' &= (\alpha^2 - \omega\beta^2)c \bmod L^3
\end{aligned}$$

for some α, β which are not both zero. Then

$$\begin{aligned}
[b', a', a', a'] &= \pm(\alpha^2 - \omega\beta^2)(\alpha^2 + 2\alpha\beta\lambda + \beta^2\mu)[b, a, a, a], \\
[b', a', a', b'] &= (\alpha^2 - \omega\beta^2)(\omega\alpha\beta + \alpha^2\lambda + \omega\beta^2\lambda + \alpha\beta\mu)[b, a, a, a], \\
[b', a', b', b'] &= \pm(\alpha^2 - \omega\beta^2)(\omega^2\beta^2 + 2\omega\alpha\beta\lambda + \alpha^2\mu)[b, a, a, a].
\end{aligned}$$

To ensure that $[b', a', a', a'] \neq 0$ we require that $\alpha^2 + 2\alpha\beta\lambda + \beta^2\mu \neq 0$, and then we see that the pair (λ, μ) transforms into

$$(\lambda', \mu') = \left(\pm\frac{\omega\alpha\beta + \alpha^2\lambda + \omega\beta^2\lambda + \alpha\beta\mu}{\alpha^2 + 2\alpha\beta\lambda + \beta^2\mu}, \frac{\omega^2\beta^2 + 2\omega\alpha\beta\lambda + \alpha^2\mu}{\alpha^2 + 2\alpha\beta\lambda + \beta^2\mu}\right)$$

with respect to the new generators a', b', c'. Note that the value of the pair (λ', μ') depends only on the ratio $\alpha : \beta$. If $\beta = 0$ we have $(\lambda', \mu') = (\pm\lambda, \mu)$. If $\beta \neq 0$ then we set $\alpha/\beta = r$, and then

$$(\lambda', \mu') = \left(\pm\frac{r^2\lambda + r(\omega + \mu) + \omega\lambda}{r^2 + 2r\lambda + \mu}, \frac{r^2\mu + 2r\omega\lambda + \omega^2}{r^2 + 2r\lambda + \mu}\right).$$

We can think of the case $\beta = 0$ as corresponding to $r = \infty$.

We now describe the orbits of the pairs (λ, μ) under this group of transformations, and show that there are $p+1$ orbits. Thus there are $p+1$ algebras with presentations of this form.

First consider the case when $\mu = \omega$. Then

$$\frac{r^2\mu + 2r\omega\lambda + \omega^2}{r^2 + 2r\lambda + \mu} = \mu$$

for all values of r, and so the orbit of (λ, ω) consists of all pairs (λ', ω) where

$$\lambda' = \pm\frac{r^2\lambda + 2r\omega + \omega\lambda}{r^2 + 2r\lambda + \omega}.$$

First note that if $r = -\lambda$ then

$$\frac{r^2\lambda + 2r\omega + \omega\lambda}{r^2 + 2r\lambda + \omega} = -\lambda,$$

so we only need to consider values of

$$\frac{r^2\lambda + 2r\omega + \omega\lambda}{r^2 + 2r\lambda + \omega}.$$

Next note that

$$\frac{r^2\lambda + 2r\omega + \omega\lambda}{r^2 + 2r\lambda + \omega} = \lambda$$

if and only if $r = 0$. So each value of λ' in the orbit occurs twice as r ranges over the $p+1$ values $0, 1, \ldots, p-1, \infty$. However if $r^2 + 2r\lambda + \omega = 0$ for any value of r then $\lambda' = \infty$ also occurs twice, and we need to discount this value. Note that $r^2 + 2r\lambda + \omega = 0$ for some value of r if and only if $\lambda^2 - \omega$ is a square. So there are two orbits of pairs (λ, ω), one of size $(p-1)/2$ and one of size $(p+1)/2$. We can obtain representatives for the two orbits by picking two values of λ, one where $\lambda^2 - \omega$ is a square, and one where it is not.

Next consider the case when $\mu = -\omega$. The pair $(0, -\omega)$ is in an orbit of size 1. If $\lambda \neq 0$ and $\mu = -\omega$ then

$$\frac{r^2\mu + 2r\omega\lambda + \omega^2}{r^2 + 2r\lambda + \mu} = \mu$$

if and only if $r = 0$ or ∞, and $r = 0$ maps $(\lambda, -\omega)$ to $(-\lambda, -\omega)$. So the orbit has $p+1$ elements if $\lambda^2 - \mu$ is not a square, and $p-1$ elements if it is a square.

And now consider the orbit of (λ, μ) where $\mu \neq \pm\omega$. Then

$$\frac{r^2\mu + 2r\omega\lambda + \omega^2}{r^2 + 2r\lambda + \mu} = \mu$$

implies that $r = \infty$ or that $r = -(\omega + \mu)/2\lambda$ (with the second case only arising when $\lambda \neq 0$). If $\lambda \neq 0$ and $r = -(\omega + \mu)/2\lambda$ then

$$\left(\frac{r^2\lambda + r(\omega + \mu) + \omega\lambda}{r^2 + 2r\lambda + \mu}, \frac{r^2\mu + 2r\omega\lambda + \omega^2}{r^2 + 2r\lambda + \mu}\right) = (-\lambda, \mu).$$

Once again, the orbit has $p+1$ elements if $\lambda^2 - \mu$ is not a square, and $p-1$ elements if it is a square. Now there are $p(p-1)$ pairs (λ, μ) with $\lambda^2 - \mu \neq 0$, and in half the pairs $\lambda^2 - \mu$ is a square, and in half it is not. So $p(p-1)/2$ pairs in which $\lambda^2 - \mu$ is a square split up into one orbit of size $(p-1)/2$ containing elements of the form (λ, ω), and into $(p-1)/2$ orbits of size $p-1$. The $p(p-1)/2$ pairs in which $\lambda^2 - \mu$ is not a square split up into one orbit of size 1 containing $(0, -\omega)$, one orbit of size $(p+1)/2$ containing elements of the form (λ, ω), and $(p-3)/2$ orbits of size $p+1$. Note that the $p-1$ elements $(\lambda, 0)$ $(\lambda \neq 0)$ must each lie in one of the $(p-1)/2$ orbits of size $p-1$. Furthermore it follows from the calculations above that $(\lambda_1, 0)$ lies in the same orbit as $(\lambda_2, 0)$ if and only if $\lambda_1 = \pm\lambda_2$. So $\{(\lambda, 0) \mid \lambda = 1, 2, \ldots, (p-1)/2\}$ forms a complete set of representatives for these orbits. Thus we have $p+1$ algebras of the form [DW.157 ˘ DW.161]

$$\begin{aligned} \langle a, b, c \mid [b, a, a, b] &= \lambda[b, a, a, a],\ [b, a, b, b] = \mu[b, a, a, a], \\ [c, a] &= -[b, a, b],\ [c, b] = -\omega[b, a, a],\ \text{class } 4\rangle, \end{aligned} \tag{7.107}$$

with $\lambda^2 - \mu \neq 0$. The algebras are deøned by the following pairs (λ, μ):

1. $(0, -\omega)$,

2. (λ, ω) where $\lambda^2 - \omega$ is a square (all these algebras are isomorphic),

3. (λ, ω) where $\lambda^2 - \omega$ is not a square (all these algebras are isomorphic),

4. $(\lambda, 0)$ where $1 \leq \lambda \leq (p-1)/2$ (giving $(p-1)/2$ diœerent algebras),

5. (λ, μ) where $\lambda^2 - \mu$ is not a square, $\mu \neq \omega$, $\lambda \neq 0$ if $\mu = -\omega$; these parameters give $(p-3)/2$ diœerent algebras with two pairs (λ, μ), (λ', μ') giving isomorphic algebras if $(\lambda, \mu) = (\lambda', \mu')$ or if

$$(\lambda', \mu') = \left(\frac{r^2\lambda + r(\omega + \mu) + \omega\lambda}{r^2 + 2r\lambda + \mu}, \frac{r^2\mu + 2r\omega\lambda + \omega^2}{r^2 + 2r\lambda + \mu} \right)$$

for some $r \in \mathbb{Z}_p$.

Note that 7.106 has a centre of dimension 2, whereas all the diœerent parameters in 7.107 listed above give algebras with centres of dimension 1. (Actually, 7.106 is 7.107 with $\lambda = \mu = 0$.) The fact that the diœerent sets of parameters (λ, μ) listed for 7.107 give diœerent algebras follows from our complete analysis of all possible generating sets a', b', c' for L which satisfy the relation of L in a, b, c modulo L^4.

5.5.7 case 7

Now let L be an immediate descendant of 6.16. Then L is generated by a, b, c, L^2 is spanned modulo L^3 by $[b, a]$, $[c, a]$, L^3 is spanned modulo L^4 by $[b, a, b, b]$. The commutators $[c, b]$, $[b, a, a]$, $[b, a, c] = [c, a, b]$, $[c, a, a]$, $[c, a, c]$ are all scalar multiples of $[b, a, b, b]$. First, replacing c by $c + \lambda[b, a, b]$ for suitable λ, we may suppose that $[c, b] = 0$.

If $[c, a]$ is central then by scaling a we have [DW.60, DW.61]

$$\langle a, b, c \,|\, [c, b] = [b, a, a] = [c, a, a] = [c, a, b] = [c, a, c] = 0, \text{ class } 4 \rangle, \tag{7.108}$$

$$\langle a, b, c \,|\, [c, b] = [c, a, a] = [c, a, b] = [c, a, c] = 0, [b, a, a] = [b, a, b, b], \text{ class } 4 \rangle. \tag{7.109}$$

Now suppose that $[c, a]$ is not central.

If $[c, a]$ is not centralized by c then replacing a and b by $a + \alpha c$, $b + \beta c$ for suitable α, β we can assume that $[c, a] = [c, b] = 0$, and by scaling b and c we can suppose that $[c, a, c] = [b, a, b, b]$. Then by scaling a we can assume that $[b, a, a] = 0$ or $[b, a, b, b]$. So we have [DW.140, DW.141]

$$\langle a, b, c \,|\, [c, b] = [b, a, a] = [c, a, a] = [c, a, b] = 0, [c, a, c] = [b, a, b, b], \text{ class } 4 \rangle, \tag{7.110}$$

$$\langle a, b, c \,|\, [c, b] = [c, a, a] = [c, a, b] = 0, [b, a, a] = [c, a, c] = [b, a, b, b], \text{ class } 4 \rangle. \tag{7.111}$$

Finally, if $[c, a]$ is centralized by c, then it must also be centralized by some non-trivial linear combination of a and b. By scaling we may suppose that $[c, a]$ is centralized by a or b or $a-b$. First suppose that c is centralized by a. So $[c,a,a]=[c,a,c]=0$, and by scaling we may suppose that $[b,a,c]=[c,a,b]=[b,a,b,b]$. Replacing a by $a+\alpha c$ for suitable α we may suppose that $[b,a,a]=0$. This gives [DW.98]

$$\langle a,b,c\,|\,[c,b]=[b,a,a]=[c,a,a]=[c,a,c]=0,\ [c,a,b]=[b,a,b,b],\ \text{class } 4\rangle. \quad (7.112)$$

And if c is centralized by b then we have $[c,a,b]=[c,a,c]=0$, and by scaling we have $[c,a,a]=[b,a,b,b]$. Replacing b by $b+\beta c$ for suitable β we may suppose that $[b,a,a]=0$. This gives [DW.142]

$$\langle a,b,c\,|\,[c,b]=[b,a,a]=[c,a,b]=[c,a,c]=0,\ [c,a,a]=[b,a,b,b],\ \text{class } 4\rangle. \quad (7.113)$$

Finally, if $[c,a]$ is centralized by $a-b$ then we have $[c,a,c]=0$ and by scaling we have

$$[c,a,a]=[c,a,b]=[b,a,c]=[b,a,b,b].$$

If we let $a'=a+\alpha c$ then

$$[c,a',a']=[c,a',b]=[b,a',c]=[b,a',b,b]=[b,a,b,b],$$

and

$$[b,a',a']=[b,a,a]+\alpha[b,a,b,b].$$

So replacing a by a' for suitable α we obtain [DW.143]

$$\langle a,b,c\,|\,[c,b]=[b,a,a]=[c,a,c]=0,\ [c,a,a]=[c,a,b]=[b,a,b,b],\ \text{class } 4\rangle. \quad (7.114)$$

To see that these seven algebras are distinct we note that $C=\langle c\rangle+L^2$ is the third term of the ascending central series, so that it is a characteristic subalgebra, and that $[C,L,C]=\{0\}$ if and only if $[c,a,c]=0$. So 7.110 and 7.111 are distinct from the other øve algebras. They are also distinct from each other since the centralizer of the derived algebra has dimension 5 in 7.110, but dimension 4 in 7.111. The centralizer of the derived algebra also has dimension 4 in 7.114, dimension 5 in 7.109, 7.112 and 7.113, and dimension 6 in 7.108. also, the centre of the algebra has dimension 2 in 7.109 and 7.112, but dimension 1 in 7.113. So it remains to distinguish 7.109 and 7.112 from each other. We ønally note that $[C,L^2]=\{0\}$ in 7.109, but that $[C,L^2]=L^4$ in 7.112.

5.5.8 case 8

Let L be an immediate descendant of 6.20. Then L is generated by a,b,c, L^2 is spanned modulo L^3 by $[b,a]$, $[c,a]$, and L^3 is spanned by $[b,a,a]$ modulo L^4 and L^4 is spanned by $[b,a,a,a]$. The commutators

$$[c,b],\ [b,a,b],\ [b,a,c]=[c,a,b],\ [c,a,a],\ [c,a,c]$$

are all linear multiples of $[b,a,a,a]$. First note that replacing c by $c+\lambda[b,a]$ for suitable λ we may suppose that $[c,a,a]=0$.

First consider the case when $[c,a]$ is central. Then by scaling b we can suppose that $[b,a,b]=0$ or $[b,a,a,a]$, and by scaling c we can suppose that $[c,b]=0$ or $[b,a,a,a]$. So we have four algebras [DW.64, DW.65, DW.67, DW.68]

$$\langle a,b,c\,|\,[c,b]=[b,a,b]=[b,a,c]=[c,a,a]=[c,a,c]=0,\ \text{class }4\rangle, \tag{7.115}$$

$$\langle a,b,c\,|\,[c,b]=[b,a,c]=[c,a,a]=[c,a,c]=0,\ [b,a,b]=[b,a,a,a],\ \text{class }4\rangle, \tag{7.116}$$

$$\langle a,b,c\,|\,[b,a,b]=[b,a,c]=[c,a,a]=[c,a,c]=0,\ [c,b]=[b,a,a,a],\ \text{class }4\rangle, \tag{7.117}$$

$$\langle a,b,c\,|\,[b,a,c]=[c,a,a]=[c,a,c]=0,\ [c,b]=[b,a,b]=[b,a,a,a],\ \text{class }4\rangle. \tag{7.118}$$

Now suppose that $[c,a]$ is not central, so that at least one of $[c,a,b]$ and $[c,a,c]$ is non-zero. Some non-trivial linear combination of b and c must centralize $[c,a]$.

If $[c,a,c]=0$ then by scaling we may assume that

$$[b,a,c]=[c,a,b]=[b,a,a,a].$$

Then replacing c by $c+\alpha[c,a]$ for suitable α we may assume that $[c,b]=0$. If we let $b'=b+\beta c$ then

$$\begin{aligned}
[b',a] &= [b,a]+\beta[c,a],\\
[b',a,a] &= [b,a,a],\\
[b',a,b'] &= [b,a,b]+2\beta[b,a,a,a],\\
[b',a,c] &= [b,a,a,a],\\
[b',a,a,a] &= [b,a,a,a],\\
[c,b'] &= 0,
\end{aligned}$$

so replacing b by b' for suitable β we may suppose that $[b,a,b]=0$. This gives [DW.152]

$$\langle a,b,c\,|\,[c,b]=[b,a,b]=[c,a,a]=[c,a,c]=0,\ [b,a,c]=[b,a,a,a],\ \text{class }4\rangle. \tag{7.119}$$

On the other hand if $[c,a,c]\neq 0$ then $b+\gamma c$ must centralize $[c,a]$ for some γ. Replacing b by $b+\gamma c$ we may suppose that $[b,a,c]=[c,a,b]=0$. Scaling b we may suppose that $[c,a,c]=[b,a,a,a]$. Now replacing b by $b+\delta[c,a]$ for suitable δ we may suppose that $[c,b]=0$. We also have $[b,a,b]=\lambda[b,a,a,a]$, and scaling a and c by the same scale factor we may suppose that $\lambda=0$, 1 or ω. This gives [DW.149, DW.150, DW.151]

$$\langle a,b,c\,|\,[c,b]=[b,a,b]=[b,a,c]=[c,a,a]=0,\ [c,a,c]=[b,a,a,a],\ \text{class }4\rangle, \tag{7.120}$$

$$\langle a,b,c\,|\,[c,b]=[b,a,c]=[c,a,a]=0,\ [b,a,b]=[c,a,c]=[b,a,a,a],\ \text{class }4\rangle, \tag{7.121}$$

$$\begin{aligned}\langle a,b,c\,|\,[c,b] &= [b,a,c]=[c,a,a]=0,\ [b,a,b]=\omega[b,a,a,a], \qquad (7.122)\\ [c,a,c] &= [b,a,a,a],\ \text{class } 4\rangle.\end{aligned}$$

To see that these eight algebras are distinct from one another ørst note that the centre has dimension 2 in 7.115 ˘ 7.118, and dimension 1 in 7.119 ˘ 7.122. Next note that the centralizer of L^2 has dimension 6 in 7.115 and 7.117, dimension 5 in 7.116, 7.118 and 7.120, and dimension 4 in 7.119, 7.121 and 7.122. In algebras 7.115 ˘ 7.118 the second term of the ascending central series is $C=\langle c,[c,a]\rangle+L^3$. In 7.115 and 7.116 the element $c\in C\backslash L^2$ has breadth 1, but in 7.117 and 7.118 every element of $C\backslash L^2$ has breadth 2. So the algebras 7.115 ˘ 7.118 are diœerent from each other as well as from 7.119 ˘ 7.122. In 7.119 ˘ 7.122 the third term of the ascending central series is $D=\langle c\rangle+L^2$. In 7.119 $[C,L,C]=\{0\}$, but in 7.120 ˘ 7.122 $[C,L,C]=L^4$. So it only remains to show that 7.121 is not isomorphic to 7.122. Now in 7.121 and 7.122 the centralizer of L^3 is the ideal $I=\langle b,c\rangle+L^2$. Consider the problem of ønding an element $d\in C\backslash L^2$ such that$[a',d,d]=0$ for all $a'\notin C$. this is equivalent to the problem of ønding scalars α,β such that

$$[a,\alpha b+\beta c,\alpha b+\beta c]=0.$$

In 7.121 we require $\alpha^2+\beta^2=0$ and in 7.122 we require $\alpha^2+\omega\beta^2=0$. Thus we can ønd these scalars in one of the two algebras, but not in the other. (Which one depends on the value of $p \bmod 4$.) So 7.121 is not isomorphic to 7.122.

5.5.9 case 9

Let L be an immediate descendant of 6.21. Then L is generated by a,b,c, L^2 is spanned modulo L^3 by $[b,a]$, $[c,a]$, and L^3 is spanned by $[b,a,a]$ modulo L^4 and L^4 is spanned by $[b,a,a,a]$. The commutators

$$[b,a,b],\ [b,a,c],\ [c,a,a],\ [c,a,c]$$

are all linear multiples of $[b,a,a,a]$. Also, $[c,b]=[b,a,a]+\alpha[b,a,a,a]$ for some α, and

$$[c,a,b]=[b,a,a,a]+[b,a,c].$$

First note that replacing c by $c+\lambda[b,a]$ for suitable λ we may suppose that $[c,a,a]=0$.

Suppose for the moment that $[c,a,c]=0$ and that $[b,a,c]=n[b,a,a,a]$ for some $n\in\mathbb{Z}_p$. Let $b'=b+\mu c$. Then

$$\begin{aligned}[b',a] &= [b,a]+\mu[c,a],\\ [c,b'] &= [c,b],\\ [b',a,a] &= [b,a,a],\\ [b',a,b'] &= [b,a,b]+\mu[c,a,b]+\mu[b,a,c]\\ &= [b,a,b]+\mu(1+2n)[b,a,a,a],\\ [b',a,c] &= n[b,a,a,a].\end{aligned}$$

So provided $n \neq -1/2$ we can replace b by b' for suitable μ so that $[b,a,b]=0$. When $n=-1/2$ then we can scale b so that $[b,a,b]=0$ or $[b,a,a,a]$. Next, let $b'=b+\nu[b,a]$. Then

$$\begin{aligned}
[b',a] &= [b,a]+\nu[b,a,a], \\
[c,b'] &= [c,b]-n\nu[b,a,a,a] \\
&= [b,a,a]+(\alpha-n\nu)[b,a,a,a], \\
[b',a,a] &= [b,a,a]+\nu[b,a,a,a], \\
[b',a,b'] &= [b,a,b], \\
[b',a,c] &= n[b,a,a,a].
\end{aligned}$$

As long as $n \neq -1$ we can choose ν so that $[c,b']=[b',a,a]$, and we replace b by b'. But if $n=-1$ then we let $a'=a+\nu c$, which gives

$$\begin{aligned}
[b,a'] &= [b,a]-\nu[b,a,a]-\nu\alpha[b,a,a,a], \\
[c,a'] &= [c,a], \\
[b,a',a'] &= [b,a,a]-2\nu[b,a,a,a], \\
[b,a',b] &= [b,a,b], \\
[b,a',c] &= [b,a,c]=-[b,a,a,a], \\
[b,a',a',a'] &= [b,a,a,a], \\
[c,a',a'] &= [c,a',c]=0,
\end{aligned}$$

so if we let $\nu=-\alpha/2$ then $[c,b]=[b,a',a']$, and we replace a by a'. So in every case we can assume that $[c,b]=[b,a,a]$, and we have the following algebras [DW.101, DW.154, DW.155]

$$\begin{aligned}
\langle a,b,c \,|\, [b,a,b] &= [c,a,a]=[c,a,c]=0,\ [c,b]=[b,a,a], \\
[b,a,c] &= n[b,a,a,a],\ \text{class } 4\rangle
\end{aligned} \tag{7.123}$$

for $n \in \mathbb{Z}_p$, together with [DW.156]

$$\begin{aligned}
\langle a,b,c \,|\, [c,a,a] &= [c,a,c]=0,\ [c,b]=[b,a,a], \\
[b,a,c] &= -\frac{1}{2}[b,a,a,a],\ [b,a,b]=[b,a,a,a],\ \text{class } 4\rangle.
\end{aligned} \tag{7.124}$$

Note that when $n=-1$ in 7.124 then $[c,a,b]=0$, so that $[c,a]$ is central which gives DW.101, and that when $n=0$ we have DW.154.

Finally, consider the case when $[c,a,c] \neq 0$. Scaling b we may suppose that $[c,a,c]=[b,a,a,a]$. Replacing b by $b+\mu c$ for suitable μ, we can assume that $[b,a,c]=0$. And then replacing (the new) b by $b+\nu[c,a]$ for suitable ν we can suppose that $[c,b]=[b,a,a]$. This gives [DW.153]

$$\begin{aligned}
\langle a,b,c \,|\, [b,a,c] &= [c,a,a]=0,\ [c,b]=[b,a,a], \\
[b,a,b] &= n[b,a,a,a],\ [c,a,c]=[b,a,a,a],\ \text{class } 4\rangle.
\end{aligned} \tag{7.125}$$

for $n \in \mathbb{Z}_p$.

In all of these algebras the third term of the ascending central series is $C = \langle c \rangle + L^2$. In 7.123 and 7.124 $[C, L, C] = \{0\}$, but in 7.125 $[C, L, C] = L^4$. We also note that the subalgbra $D = \langle b, c \rangle + L^2$ is characteristic in all these algebras since it is the centralizer of L^3.

For the algebras 7.123 and 7.124 we consider possible generating sets a', b', c' satisfying

$$[c', a', a'] = [c', a', c'] = 0,\ [c', b'] = [b', a', a'],$$

with L^4 spanned by $[b', a', a', a']$ and $[b', a', b']$, $[b', a', c'] \in L^4$. Clearly $a' \notin D$, $b' \in D \backslash C$, and $c' \in c \backslash L^2$. So

$$\begin{aligned} a' &= \alpha a + \beta b + \gamma c + u, \\ b' &= \delta b + \varepsilon c + v, \\ c' &= \eta c + w, \end{aligned}$$

for some scalars $\alpha, \beta, \gamma, \delta, \varepsilon, \eta$ with $\alpha, \delta, \eta \neq 0$ and $u, v, w \in L^2$. Then

$$\begin{aligned} [c', b'] &= \delta\eta[b, a, a] \bmod L^4, \\ [b', a', a'] &= \alpha^2\delta[b, a, a] \bmod L^4, \\ [b', a', c'] &= \alpha\delta\eta[b, a, c], \\ [b', a', a', a'] &= \alpha^3\delta[b, a, a, a]. \end{aligned}$$

Since we require $[c', b'] = [b', a', a']$, we must have $\eta = \alpha^2$, so that $[b', a', c'] = n[b', a', a', a']$. So dicerent values of n in 7.123 give dicerent algebras. Also, 7.124 could only be isomorphic to 7.123 for $n = -1/2$. In this case

$$\begin{aligned} [b', a', b'] &= \alpha\delta^2[b, a, b] + \alpha\delta\varepsilon([b, a, c] + [c, a, b]) \\ &= \alpha\delta^2[b, a, b] \end{aligned}$$

so that there no generators a', b', c' for 7.124 with $[b', a', b'] = 0$.

Similarly, for the algebras 7.125 we consider possible sets of generators a', b', c' satsifying

$$[b', a', c'] = [c', a', a'] = 0,\ [c', b'] = [b', a', a'],\ [c', a', c'] = [b', a', a', a'] \neq 0,$$

with $[b', a', b'] \in L^4$. Clearly a', b', c' must be of the same form as described above, and then

$$[b', a', c'] = \alpha\varepsilon\eta[b, a, a, a]$$

so that we must have $\varepsilon = 0$. This implies that

$$\begin{aligned} [c', b'] &= \delta\eta[b, a, a] \bmod L^4, \\ [b', a', a'] &= \alpha^2\delta[b, a, a] \bmod L^4, \\ [b', a', b'] &= \alpha\delta^2[b, a, b], \\ [b', a', a', a'] &= \alpha^3\delta[b, a, a, a], \\ [c', a', c'] &= \alpha\eta^2[b, a, a, a]. \end{aligned}$$

To ensure that $[c',b'] = [b',a',a']$ we must have $\eta = \alpha^2$, and to ensure that $[c',a',c'] = [b',a',a',a']$ we must have $\delta = \alpha^2$ so that if $[b,a,b] = n[b,a,a,a]$ then $[b',a',b'] = n[b',a',a',a']$. So dœerent values of n give dœerent algebras.

5.5.10 case 10

Let L be an immediate descendant of 6.22. Then L is generated by a, b, c, L^2 is spanned by $[b,a]$ modulo L^3, L^3 is spanned by $[b,a,b]$ modulo L^4, L^4 is spanned by $[b,a,b,b]$ modulo L^5 and L^5 is spanned by $[b,a,b,b,a]$ and $[b,a,b,b,b]$. The commutators $[b,a,a]$ and $[c,b]$ lie in L^5 and $[c,a] = -[b,a,b,b]$ modulo L^5. Since L^5 has dimension 1, $[b,a,b,b]$ must be centralized by some non-trivial linear combination of a and b, and we may suppose that $[b,a,b,b]$ is centralized by a or by b.

First suppose that $[b,a,b,b,a] = 0$, so that L^5 is spanned by $[b,a,b,b,b]$. Replacing c by $c + \lambda[b,a,b,b]$ for suitable λ we may suppose that $[c,b] = 0$. Then replacing a by $a + \mu c$ for suitable μ we may suppose that $[b,a,a] = 0$. So $[c,a] = -[b,a,b,b] + \alpha[b,a,b,b,b]$, and scaling b and c we may suppose that $\alpha = 0$ or 1. This gives [DW.165, DW.166]

$$\begin{aligned} \langle a,b,c \,|\, [c,b] &= [b,a,a] = 0,\ [c,a] = -[b,a,b,b], \\ [b,a,b,b,a] &= 0,\ \text{class } 5\rangle, \end{aligned} \tag{7.126}$$

$$\begin{aligned} \langle a,b,c \,|\, [c,b] &= [b,a,a] = 0,\ [c,a] = -[b,a,b,b] + [b,a,b,b,b], \\ [b,a,b,b,a] &= 0,\ \text{class } 5\rangle. \end{aligned} \tag{7.127}$$

Next suppose that $[b,a,b,b,b] = 0$ so that L^5 is spanned by $[b,a,b,b,a]$. Replacing c by $c + \lambda[b,a,b,b]$ for suitable λ we may suppose that $[c,a] = -[b,a,b,b]$. And replacing b by $b + \mu c$ for suitable μ we may suppose that $[b,a,a] = 0$. So $[c,b] = \alpha[b,a,b,b,a]$, and scaling we may suppose that $\alpha = 0$ or 1. This gives [DW.167, DW.168]

$$\begin{aligned} \langle a,b,c \,|\, [c,b] &= [b,a,a] = 0,\ [c,a] = -[b,a,b,b], \\ [b,a,b,b,b] &= 0,\ \text{class } 5\rangle, \end{aligned} \tag{7.128}$$

$$\begin{aligned} \langle a,b,c \,|\, [b,a,a] &= 0,\ [c,a] = -[b,a,b,b],\ [c,b] = [b,a,b,b,a], \\ [b,a,b,b,b] &= 0,\ \text{class } 5\rangle. \end{aligned} \tag{7.129}$$

Note that the centralizer of the derived algebra has dimension 5 in 7.126 and 7.127, but dimension 3 in 7.128 and 7.129. In 7.128 and 7.129 the second term of the ascending central series is $C = \langle c\rangle + L^4$. In 7.128 c has breadth 1, but no element of $C\backslash L^4$ has breadth 1 in 7.129. So 7.128 and 7.129 are dœerent from 7.126 and

7.127, as well as from each other. To show that 7.126 and 7.127 are not isomorphic we consider generating sets a', b', c' for 7.126 and 7.127 satisfying the relations

$$\begin{aligned} [c', b'] &= [b', a', a'] = [b', a', b', b', a'] = 0, \\ [c', a'] &= -[b, a, b, b] \bmod L^5, \end{aligned}$$

with $[b', a', b', b', b'] \neq 0$. Note that in 7.126 and 7.127 the third term of the ascending central series is $D = \langle c \rangle + L^3$, and since $c' \in D$ we must have $c' = \alpha c$ modulo L^3 for some $\alpha \neq 0$. So a', b' must be linearly independant modulo $\langle c \rangle + L^2$, and this implies that $[b', a'] = \beta[b, a]$ modulo L^3 for some $\beta \neq 0$. If

$$a' = \lambda a + \mu b + \nu c + \xi[b, a] \bmod L^3$$

then

$$0 = [b', a', a'] = \mu\beta[b, a, b] \bmod L^4$$

so that $\mu = 0$ and

$$0 = [b', a', a'] = \beta\nu[b, a, c] = -\beta\nu[b, a, b, b, b].$$

So $\nu = 0$ also, which imples that $\lambda \neq 0$. Now let

$$b' = \delta a + \varepsilon b + \eta c \bmod L^2,$$

where $\varepsilon \neq 0$. Then

$$\begin{aligned} [b', a'] &= \varepsilon\lambda[b, a] - \varepsilon\xi[b, a, b] \bmod L^4, \\ [b', a', b'] &= \varepsilon^2\lambda[b, a, b] - \varepsilon^2\xi[b, a, b, b] \bmod L^5, \\ [b', a', b', b'] &= \varepsilon^3\lambda[b, a, b, b] - \varepsilon^3\xi[b, a, b, b, b], \\ [b', a', b', b', b'] &= \varepsilon^4\lambda[b, a, b, b, b], \\ [c', a'] &= \alpha\lambda[c, a] - \alpha\xi[b, a, c] \\ &= \alpha\lambda[c, a] + \alpha\xi[b, a, b, b, b]. \end{aligned}$$

then we must have $\alpha = -\varepsilon^3$ so that if $[c, a] = [b, a, b, b]$ then $[c', a'] = [b', a', b', b']$. So 7.126 and 7.127 are not isomorphic.

5.5.11 case 11

Let L be an immediate descendant of 6.23. Then L is generated by a, b, c, L^2 is spanned by $[b, a]$ modulo L^3, L^3 is spanned by $[b, a, b]$ modulo L^4, L^4 is spanned by $[b, a, b, b]$ modulo L^5 and L^5 is spanned by $[b, a, b, b, a]$ and $[b, a, b, b, b]$. The commutator $[c, b]$ lies in L^5, $[b, a, a] = [b, a, b, b]$ modulo L^5 and $[c, a] = -[b, a, b, b]$ modulo L^5. Since L^5 has dimension 1, $[b, a, b, b]$ must be centralized by some non-trivial linear combination of a and b, and we may suppose that $[b, a, b, b]$ is centralized by a or by b.

First suppose that $[b,a,b,b,a]=0$, so that L^5 is spanned by $[b,a,b,b,b]$. If we replace a by $a+\alpha c$ for suitable α we may suppose that $[b,a,a]=[b,a,b,b]$. And replacing c by $c+\beta[b,a,b,b]$ for suitable β we may suppose that $[c,b]=0$. Now suppose that

$$[c,a]=-[b,a,b,b]+\lambda[b,a,b,b,b].$$

We want to show that we can assume that $\lambda=0$. Let $\gamma\in\mathbb{Z}_p$ and let

$$\begin{aligned} a' &= a-2\gamma c,\\ b' &= b+\gamma a,\\ c' &= c+\gamma[b,a,b]-(\gamma^2+\gamma\lambda)[b,a,b,b]. \end{aligned}$$

Then

$$\begin{aligned} [b',a'] &= [b,a] \bmod L^4,\\ [b',a',a'] &= [b,a,a]-2\gamma[b,a,c]\\ &= [b,a,b,b]+2\gamma[b,a,b,b,b],\\ [b',a',b'] &= [b,a,b]+\gamma[b,a,b,b] \bmod L^5,\\ [b',a',b',b'] &= [b,a,b,b]+2\gamma[b,a,b,b,b],\\ [b',a',b',b',b'] &= [b,a,b,b,b],\\ [c',b'] &= \gamma[c,a]+\gamma[b,a,b,b]-\gamma\lambda[b,a,b,b,b]=0,\\ [c',a'] &= [c,a]+\gamma[b,a,b,b,b]\\ &= -[b,a,b,b]+(\gamma+\lambda)[b,a,b,b,b]. \end{aligned}$$

So $[b',a',a']=[b',a',b',b']$, $[c',b']=0$, and $[c',a']=-[b',a',b',b']$ provided $\gamma=-\lambda/3$. So replacing a,b,c by a',b',c' for this value of γ we obtain [DW.173]

$$\begin{aligned} \langle a,b,c\,|\,[b,a,a] &= [b,a,b,b],\ [c,a]=-[b,a,b,b],\ [c,b]=0,\\ [b,a,b,b,a] &= 0,\ \text{class } 5\rangle. \end{aligned} \tag{7.130}$$

In the case when $p=3$ we cannot take $\lambda=0$, but if $\lambda\neq 0$, then we can take $\lambda=1$ by scaling. This gives

$$\begin{aligned} \langle a,b,c\,|\,[b,a,a] &= [b,a,b,b],\ [c,a]=-[b,a,b,b]+[b,a,b,b,b],\\ [c,b] &= 0,\ [b,a,b,b,a]=0,\ \text{class } 5\rangle. \end{aligned} \tag{7.130A}$$

Next, suppose that $[b,a,b,b,b]=0$, so that L^5 is spanned by $[b,a,b,b,a]$. Replacing c by $c+\alpha[b,a,b,b]$ for suitable α we may suppose that $[c,a]=-[b,a,b,b]$. And replacing b by $b+\beta c$ for suitable β, we may suppose that $[b,a,a]=[b,a,b,b]$. We also have $[c,b]=\gamma[b,a,b,b,a]$ for some γ. Setting $a'=\lambda^2 a$, $b'=\lambda b$, $c'=\lambda^3 c$ $(\lambda\neq 0)$ we have

$$\begin{aligned} [c',a'] &= \lambda^5[c,a]=-[b',a',b',b'],\\ [b',a',a'] &= \lambda^5[b,a,a]=[b',a',b',b'],\\ [c',b'] &= \lambda^4[c,b]=(\gamma.\lambda^{-3})[b',a',b',b',a']. \end{aligned}$$

So we may assume that $\gamma = 0$, or 1, or (when $p = 1 \bmod 3$) ω or ω^2. This gives [DW.169 ˘ DW.172]

$$\begin{aligned} \langle a, b, c \mid [b, a, a] &= [b, a, b, b],\ [c, a] = -[b, a, b, b],\ [b, a, b, b, b] = 0, \\ [c, b] &= \gamma[b, a, b, b, a],\ \text{class } 5\rangle. \end{aligned} \tag{7.131}$$

In 7.130 the centraizer of the derived algebra has dimension 4, but in 7.131 it has dimension 3, so these algebras are diœerent. In 7.131 the third term of the ascending central series is $\langle c\rangle + L^3$, and the centralizer of L^4 is $\langle b, c\rangle + L^2$. Also, the inverse image in L of the centraizer of L^3/L^5 is $\langle a, c\rangle + L^2$. So if a', b', c' are any generators of 7.131 which satisfy the relations

$$[b', a', a'] = [b', a', b', b'],\ [c', a'] = -[b', a', b', b'],\ [b', a', b', b', b'] = 0,$$

with L^5 spanned by $[b', a', b', b', a']$, and with $[c', b'] \in L^5$ then

$$\begin{aligned} a' &= \alpha a + \beta c + \delta[b, a] \bmod L^3 \\ b' &= \mu b + \nu c + \xi[b, a] \bmod L^3, \\ c' &= \rho c + \sigma[b, a, b] \bmod L^4, \end{aligned}$$

for some $\alpha, \beta, \delta, \mu, \nu, \rho, \sigma$ with $\alpha, \mu, \rho \neq 0$. So

$$[c', b'] = \mu\rho[c, b] + \mu\sigma[b, a, b, b] + \xi\sigma[b, a, b, b, a].$$

This implies that $\sigma = 0$ and that $[c', b'] = \mu\rho[c, b]$. We also have

$$\begin{aligned} [b', a', a'] &= \alpha^2\mu[b, a, a] \bmod L^5, \\ [c', a'] &= \alpha\rho[c, a] \bmod L^5, \\ [b', a', b', b'] &= \alpha\mu^3[b, a, b, b] \bmod L^5, \\ [b', a', b', b', a'] &= \alpha^2\mu^3[b, a, b, b, a]. \end{aligned}$$

So $\alpha^2\mu = \alpha\rho = \alpha\mu^3$ so that $\alpha = \mu^2$, $\rho = \mu^3$ and

$$[c', b'] = \mu^4[c, b] = \mu^4\gamma[b, a, b, b, a] = (\gamma.\mu^{-3})[b', a', b', b', a'].$$

So the diœerent values of γ speciøed in 7.131 give diœerent algebras.

5.5.12 case 12

Let L be an immediate descendant of 6.24. Then L is generated by a, b, c, and L has class 5. The subalgebra A generated by a, b is an immediate descendant of 5.8, and has dimension 6. So A is isomorphic to to 6.30, 6.31 or 6.32. Also $[c, a]$, $[c, b] \in L^5$. We also have L^4 is spanned by $[b, a, b, b]$, with L^5 spanned by $[b, a, b, b, b]$ when A is 6.30 or 6.31, and L^5 spanned by $[b, a, b, b, a]$ in 6.32. So replacing c by $c + \alpha[b, a, b, b]$

for suitable α we may assume that $[c,b]=0$ when A is 6.30 or 6.31, and that $[c,a]=0$ when A is 6.32. Thus we get six algebras [DW.30 ˘ DW.32, DW.121 ˘ DW.123]

$$\langle a,b \mid [b,a,a]=[b,a,b,b,a]=0,\ \text{class } 5\rangle \oplus \langle c\rangle, \tag{7.132}$$

$$\langle a,b \mid [b,a,a]=[b,a,b,b,b],\ [b,a,b,b,a]=0,\ \text{class } 5\rangle \oplus \langle c\rangle, \tag{7.133}$$

$$\langle a,b \mid [b,a,a]=[b,a,b,b,b]=0,\ \text{class } 5\rangle \oplus \langle c\rangle, \tag{7.134}$$

$$\langle a,b,c \mid [b,a,a]=[b,a,b,b,a]=[c,b]=0,\ [c,a]=[b,a,b,b,b],\ \text{class } 5\rangle, \tag{7.135}$$

$$\langle a,b,c \mid [b,a,a]=[c,a]=[b,a,b,b,b],\ [b,a,b,b,a]=[c,b]=0,\ \text{class } 5\rangle, \tag{7.136}$$

$$\langle a,b,c \mid [b,a,a]=[b,a,b,b,b]=[c,a]=0,\ [c,b]=[b,a,b,b,a],\ \text{class } 5\rangle. \tag{7.137}$$

Note that the ørst three of these algebras have centres of dimension 2, and that the last three have centres of dimension 1. Also, in each of the six algebras any 2-generator six dimensional subalgebra is isomorphic to A, so these six algebras are all diœerent.

5.5.13 case 13

Let L be an immediate descendant of 6.25. Then L is generated by a,b,c, and L has class 5. The subalgebra A generated by a,b is an immediate descendant of 5.9, and has dimension 6. So A is isomorphic to to 6.33 or 6.34, and just as in case 12 above we obtain four distinct algebras [DW.33, DW.34, DW.124, DW.125]

$$\langle a,b \mid [b,a,a]=[b,a,b,b],\ [b,a,b,b,a]=0,\ \text{class } 5\rangle \oplus \langle c\rangle, \tag{7.138}$$

$$\langle a,b \mid [b,a,a]=[b,a,b,b],\ [b,a,b,b,b]=0,\ \text{class } 5\rangle \oplus \langle c\rangle, \tag{7.139}$$

$$\begin{aligned}\langle a,b,c \mid [b,a,a] &= [b,a,b,b],\ [b,a,b,b,a]=[c,b]=0,\\ [c,a] &= [b,a,b,b,b],\ \text{class } 5\rangle,\end{aligned} \tag{7.140}$$

$$\begin{aligned}\langle a,b,c \mid [b,a,a] &= [b,a,b,b],\ [b,a,b,b,b]=[c,a]=0,\\ [c,b] &= [b,a,b,b,a],\ \text{class } 5\rangle.\end{aligned} \tag{7.141}$$

5.5.14 case 14

Let L be an immediate descendant of 6.26. Then L is generated by a,b,c, L^2 is spanned modulo L^3 by $[b,a]$, L^3 is spanned modulo L^4 by $[b,a,b]$, L^4 is spanned modulo L^5 by $[b,a,b,b]$, and L^5 is spanned by $[b,a,b,b,b]$. The commutators $[c,b]$ and $[b,a,a]$ lie in L^5, and $[c,a]=-[b,a,b]+\alpha[b,a,b,b,b]$ for some α. Replacing c by $c+\lambda[b,a,b,b]$ for suitable λ we can assume that $[c,b]=0$. By scaling we may assume that $[b,a,a]=0$ or $[b,a,b,b,b]$.

First consider the case when $[b,a,a]=0$. By scaling we can assume that $\alpha=0,1$ or ω. This gives [DW.186, DW.187, DW.188, though the parameters for DW.187, DW.188 seem weird! They work for $p=7$, but for $p=11$ I reckon that $n=1$ in DW.187 and $m=3$ in DW.188, and these give isomorphic groups.]

$$\langle a,b,c \mid [b,a,a]=[c,b]=0,\ [c,a]=-[b,a,b]+\alpha[b,a,b,b,b],\ \text{class } 5\rangle\ (\alpha=0,1,\omega). \tag{7.142}$$

On the other hand, when $[b,a,a]=[b,a,b,b,b]$, then let $a'=a$, $b'=b+\mu a$, $c'=c+\mu[b,a]+\nu[b,a,b,b]$. Then

$$\begin{aligned}
[b',a'] &= [b,a],\\
[c',a'] &= [c,a]+\mu[b,a,b,b,b],\\
[c',b'] &= \mu[c,a]+\mu[b,a,b]+(\mu^2+\nu)[b,a,b,b,b],\\
[b',a',a'] &= [b,a,a]=[b,a,b,b,b],\\
[b',a',b'] &= [b,a,b]+\mu[b,a,b,b,b],\\
[b',a',b',b'] &= [b,a,b,b],\\
[b',a',b',b',b'] &= [b,a,b,b,b].
\end{aligned}$$

So, by suitable choice of μ and ν we can ensure that

$$\begin{aligned}
[c',b'] &= 0,\\
[b',a',a'] &= [b',a',b',b',b'],\\
[c',a'] &= -[b',a',b']
\end{aligned}$$

and replacing a,b,c by a',b',c' we have [DW.189]

$$\langle a,b,c \mid [c,b]=0,\ [c,a]=-[b,a,b],[b,a,a]=[b,a,b,b,b],\ \text{class } 5\rangle. \tag{7.143}$$

The centralizer of L^2 has dimension 4 in 7.143, but dimension 5 in 7.142. So 7.142 is diœerent from 7.143. To show that we get diœerent algebras in 7.142 for the three diœerent values of α we consider possible generating sets a',b',c' for 7.142 satisfying the relations

$$[b',a',a']=[c',b']=0,\ [c',a']=-[b',a',b'] \bmod L^5,$$

with L^5 spanned by $[b',a',b',b',b']$. First note that if a',b',c' satisfy these relations then

$$a'\in C_L(L^2)=\langle a\rangle+L^2,$$

and c' lies in the inverse image in L of the centre of L/L^3, $\langle c\rangle+L^2$, and so

$$\begin{aligned}
a' &= \beta a+u,\\
b' &= \gamma a+\delta b+\varepsilon c+v,\\
c' &= \eta c+w,
\end{aligned}$$

for some $\beta, \gamma, \delta, \varepsilon, \eta$ with $\beta, \delta, \eta \neq 0$ and $u, v, w \in L^2$. Using the fact that if $u \in L^2$ then $[c, u] = [u, b, b]$ we see that

$$\begin{aligned}
[b', a'] &= \beta\delta[b, a] - \delta[u, b] + \beta\varepsilon[c, a] + \varepsilon[u, b, b], \\
[c', a'] &= \beta\eta[c, a] + \eta[u, b, b] \\
&= -\beta\eta[b, a, b] + \alpha\beta\eta[b, a, b, b, b] + \eta[u, b, b], \\
[b', a', b'] &= \beta\delta^2[b, a, b] - \delta^2[u, b, b] + \beta\delta\varepsilon[c, a, b] + \varepsilon[u, b, b, b] \\
&\quad +\beta\delta\varepsilon[b, a, c] - \delta\varepsilon[u, b, c] + \beta\varepsilon^2[c, a, c] \\
&= \beta\delta^2[b, a, b] - \delta^2[u, b, b] - \beta\delta\varepsilon[b, a, b, b] + \varepsilon[u, b, b, b] \\
&\quad -\beta\delta\varepsilon[b, a, b, b] + \delta\varepsilon[u, b, b, b] + \beta\varepsilon^2[b, a, b, b, b].
\end{aligned}$$

Now we require $[c', a'] = -[b', a', b']$ modulo L^5, and this implies that $\eta = \delta^2$ and $\varepsilon = 0$. So

$$\begin{aligned}
[c', a'] &= -\beta\delta^2[b, a, b] + \alpha\beta\delta^2[b, a, b, b, b] + \delta^2[u, b, b], \\
[b', a', b'] &= \beta\delta^2[b, a, b] - \delta^2[u, b, b], \\
[b', a', b', b'] &= \beta\delta^3[b, a, b, b] - \delta^3[u, b, b, b], \\
[b', a', b', b', b'] &= \beta\delta^4[b, a, b, b, b].
\end{aligned}$$

This implies that

$$[c', a'] = -[b', a', b'] + \alpha\delta^{-2}[b', a', b', b', b'],$$

and so the values $0, 1, \omega$ for α do indeed give diœerent algebras.

5.6 2 generators

If L is a two generator algebra of dimension 7 then L is an immediate descendant of one of 5.7 ˘ 5.9, 6.27 ˘ 6.31, or 6.33. (The algebras 6.32 and 6.34 are terminal.)

5.6.1 case 1

Let L be a descendant of 5.7. Then L is generated by a, b, L/L^4 is free of rank 2, and L^4 is spanned by $[b, a, a, a]$, $[b, a, a, b]$ and $[b, a, b, b]$ and has dimension 2. So L must satisfy a relation

$$\alpha[b, a, a, a] + \beta[b, a, a, b] + \gamma[b, a, b, b] = 0.$$

If $\alpha = \gamma = 0$ then we have a relation $[b, a, a, b] = 0$, and we replace a by $a + b$, b by $a - b$, which gives us the relation

$$[b, a, a, a] - [b, a, b, b] = 0.$$

So we can assume that at least one of α, γ is non-zero, and swapping a and b if necessary, we may assume that $\alpha = 1$. If we now replace a by $a + (\beta/2)b$ we get the relation

$$[b, a, a, a] = (\beta^2/4 - \gamma)[b, a, b, b].$$

After scaling this gives us three algebras [DW.96, DW.95, DW.97]

$$\langle a, b \mid [b,a,a,a] = \lambda[b,a,b,b],\ \text{class } 4\rangle\ (\lambda = 0, 1, \omega). \tag{7.144}$$

To show that these three algebras are dœerent from each other, we count elements of breadth 1. Clearly any element of breadth 1 must lie in L^3. So consider an element $u = \alpha[b,a,a] + \beta[b,a,b] + g$ where $g \in L^4$. If $[b,a,a,a] = 0$ then we must have $\alpha \neq 0$, $\beta = 0$ for u to have breadth 1. So when $\lambda = 0$ there are $p^3 - p^2$ elements of breadth 1. If $\lambda = 1$ then

$$\begin{aligned} [u,a] &= \beta[b,a,a,b] + \alpha[b,a,b,b], \\ [u,b] &= \alpha[b,a,a,a] + \beta[b,a,b,b], \end{aligned}$$

so that u has breadth 1 if $\alpha = \pm\beta \neq 0$. So when $\lambda = 1$ there are $2(p^3 - p^2)$ elements of breadth 1. Finally, if $\lambda = \omega$ then

$$\begin{aligned} [u,a] &= \beta[b,a,a,b] + \alpha\omega[b,a,b,b], \\ [u,b] &= \alpha[b,a,a,b] + \beta[b,a,b,b], \end{aligned}$$

and there are no elements of breadth 1, since $\beta^2 - \alpha^2\omega = 0$ only if $\alpha = \beta = 0$.

5.6.2 case 2

Let L be a descendant of 5.8. Then L is generated by a, b, L^2 is spanned modulo L^3 by $[b,a]$, L^3 is spanned modulo L^4 by $[b,a,b]$, L^4 is spanned modulo L^5 by $[b,a,b,b]$, and L^5 has dimension 2, and is spanned by $[b,a,b,b,a]$ and $[b,a,b,b,b]$. We have $[b,a,a] \in L^5$. Replacing a by $a + \alpha[b,a,b]$ for suitable α, we may suppose that

$$[b,a,a] = \lambda[b,a,b,b,b],$$

where by scaling we may take $\lambda = 0$ or 1. This gives [DW.105, DW.106]

$$\langle a, b \mid [b,a,a] = \lambda[b,a,b,b,b],\ \text{class } 5\rangle\ (\lambda = 0, 1). \tag{7.145}$$

We distinguish between these two algebras by noting that if $\lambda = 0$ then a has breadth 2, but if $\lambda = 1$ then every element outside L^2 has breadth at least 3.

5.6.3 case 3

Let L be a descendant of 5.9. Then L is generated by a, b, L^2 is spanned modulo L^3 by $[b,a]$, L^3 is spanned modulo L^4 by $[b,a,b]$, L^4 is spanned modulo L^5 by $[b,a,b,b]$, and L^5 has dimension 2, and is spanned by $[b,a,b,b,a]$ and $[b,a,b,b,b]$. We have

$$[b,a,a] = [b,a,b,b] + \alpha[b,a,b,b,a] + \beta[b,a,b,b,b]$$

for some α, β. Let $a' = a + \lambda[b,a,b]$ and let $b' = b + \mu a$. Then

$$\begin{aligned}
[b', a'] &= [b,a] - \lambda[b,a,b,b] - \lambda\mu[b,a,b,b,b], \\
[b', a', a'] &= [b,a,a] - 2\lambda[b,a,b,b,a] \\
&= [b,a,b,b] + (\alpha - 2\lambda)[b,a,b,b,a] + \beta[b,a,b,b,b], \\
[b', a', b'] &= [b,a,b] + \mu[b,a,b,b] \bmod L^5, \\
[b', a', b', b'] &= [b,a,b,b] + \mu^2[b,a,b,b,a] + 2\mu[b,a,b,b,b], \\
[b', a', b', b', a'] &= [b,a,b,b,a], \\
[b', a', b', b', b'] &= \mu[b,a,b,b,a] + [b,a,b,b,b].
\end{aligned}$$

So we can choose λ, μ so that $[b', a', a'] = [b', a', b', b']$, and this gives us [DW.109]

$$\langle a, b \mid [b,a,a] = [b,a,b,b], \text{ class } 5\rangle. \quad (7.146)$$

5.6.4 case 4

Let L be a descendant of 6.27. Then L is generated by a, b, L/L^4 is free of class 3, L^4 is spanned modulo L^5 by $[b,a,b,b]$ and L^5 is spanned by $[b,a,b,b,a]$ and $[b,a,b,b,b]$. The commutators $[b,a,a,a]$ and $[b,a,a,b]$ both lie in L^5. Since L^5 has dimension 1, $[b,a,b,b,a]$ and $[b,a,b,b,b]$ must be linearly dependant.

First consider the case when $[b,a,b,b,a] = 0$. Then $[b,a,a,a]$ and $[b,a,a,b]$ are linear multiples of $[b,a,b,b,b]$. By scaling we can suppose that $[b,a,a,a] = 0$ or $[b,a,b,b,b]$.

If $[b,a,a,a] = 0$. Then by scaling we may suppose that $[b,a,a,b] = 0$ or $[b,a,b,b,b]$. This gives [DW.103, DW.107]

$$\langle a, b \mid [b,a,a,a] = [b,a,a,b] = [b,a,b,b,a] = 0, \text{ class } 5\rangle, \quad (7.147)$$

$$\langle a, b \mid [b,a,a,a] = [b,a,b,b,a] = 0,\ [b,a,a,b] = [b,a,b,b,b], \text{ class } 5\rangle. \quad (7.148)$$

On the other hand, if $[b,a,a,a] = [b,a,b,b,b]$ then replacing b by $b + \lambda a$ for suitable λ, we may suppose that $[b,a,a,b] = 0$. This gives [DW.162]

$$\langle a, b \mid [b,a,a,b] = [b,a,b,b,a] = 0,\ [b,a,a,a] = [b,a,b,b,b], \text{ class } 5\rangle. \quad (7.149)$$

Next, consider the case when $[b,a,b,b,a]$ is non-zero, and $[b,a,b,b,b] = \alpha[b,a,b,b,a]$. Replacing b by $b - \alpha a$ we have $[b,a,b,b,b] = 0$. Then we have $[b,a,a,a] = \lambda[b,a,b,b,a]$, $[b,a,a,b] = \mu[b,a,b,b,a]$ for some λ, μ. By scaling b we can take $\mu = 0$ or 1, and then scaling a we can take $\lambda = 1$ or 1. So we get four algebras [DW.104, DW.108, DW.163, DW.164]

$$\langle a, b \mid [b,a,a,a] = [b,a,a,b] = [b,a,b,b,b] = 0, \text{ class } 5\rangle, \quad (7.150)$$

$$\langle a, b \mid [b,a,a,b] = [b,a,b,b,b] = 0,\ [b,a,a,a] = [b,a,b,b,a], \text{ class } 5\rangle, \quad (7.151)$$

$$\langle a, b \mid [b,a,a,a] = [b,a,b,b,b] = 0,\ [b,a,a,b] = [b,a,b,b,a], \text{ class } 5\rangle, \quad (7.152)$$

$$\langle a, b \mid [b,a,b,b,b] = 0,\ [b,a,a,a] = [b,a,a,b] = [b,a,b,b,a],\ \text{class } 5\rangle. \qquad (7.153)$$

Note that the centres of 7.147, 7.148, 7.150, 7.151 all have dimension 2 (they contain either $[b,a,a]$ or $[b,a,a] - [b,a,b,b]$), but that the other three algebras have centres of dimension 1. Also note that L is metabelian in 7.147 ˘ 7.149, but not in the other four algebras.

We distinguish between 7.147 and 7.148 by counting elements of breadth 1. Clearly any element of breadth 1 must lie in L^3. In 7.147 there are $p^2(p^2-1)$ elements of breadth 1, but in 7.148 there are only $p^2(p-1)$ elements of breadth 1.

In 7.150, $[b,a,a] \in \zeta(L)\backslash L^4$, but 7.151 has no element $[b',a',a'] \in \zeta(L)\backslash L^4$. And in 7.152, $[b,a,a,a] = 0$ but 7.153 has no elements a', b' with $[b',a',a'] \notin L^4$, $[b',a',a',L] \leq L^5$, $[b',a',a',a'] = 0$.

5.6.5 case 5

Let L be a descendant of 6.28. Then L is generated by a, b, L/L^4 is free of class 3, L^4 is spanned modulo L^5 by $[b,a,b,b]$ and L^5 is spanned by $[b,a,a,a,a]$ and $[b,a,a,a,b]$. The commutator $[b,a,a,b]$ lies in L^5, and $[b,a,b,b] = [b,a,a,a]$ modulo L^5. Since L^5 has dimension 1, $[b,a,a,a,a]$ and $[b,a,a,a,b]$ must be linearly dependant.

We consider all possible pairs of generators a', b' such that

$$\begin{aligned} [b',a',a',b'] &\in L^5, \\ [b',a',b',b'] &= [b',a',a',a'] \bmod L^5. \end{aligned}$$

It is straightforward to show that

$$\begin{aligned} a' &= \alpha a + \beta b \bmod L^2, \\ b' &= \pm(\beta a - \alpha b) \bmod L^2, \end{aligned}$$

for some α, β with $\alpha^2 + \beta^2 \neq 0$. We then obtain

$$\begin{aligned} [b',a',b',b'] - [b',a',a',a'] &= \pm\left(-(\alpha^4-\beta^4)([b,a,b,b]-[b,a,a,a]) + 4\alpha\beta(\alpha^2+\beta^2)[b,a,a,b]\right), \\ [b',a',a',b'] &= \alpha\beta(\alpha^2+\beta^2)([b,a,b,b]-[b,a,a,a]) + (\alpha^4-\beta^4)[b,a,a,b], \\ [b',a',a',a',a'] &= \pm\left(-\alpha(\alpha^2+\beta^2)^2[b,a,a,a,a] - \beta(\alpha^2+\beta^2)^2[b,a,a,a,b]\right), \\ [b',a',a',a',b'] &= -\beta(\alpha^2+\beta^2)^2[b,a,a,a,a] + \alpha(\alpha^2+\beta^2)^2[b,a,a,a,b]. \end{aligned}$$

So if we have a relation

$$\alpha[b,a,a,a,a] + \beta[b,a,a,a,b] = 0,$$

then provided $\alpha^2 + \beta^2 \neq 0$ then we can replace a, b by $\alpha a + \beta b, \beta a - \alpha b$ and we then have $[b,a,a,a,a] = 0$. Note that $\alpha^2 + \beta^2 = 0$ can only occur if $p = 1 \bmod 4$, and we deal with this case later.

So we assume that $[b,a,a,a,a]=0$, and that

$$\begin{aligned}[b,a,a,b] &= \lambda[b,a,a,a,b],\\ [b,a,b,b]-[b,a,a,a] &= \mu[b,a,a,a,b]\end{aligned}$$

for some λ,μ. By scaling we may assume that $\lambda=0$ or 1. If $\lambda=0$ then we can assume that $\mu=0$ or 1 by scaling, and this gives [DW.182, DW.181]

$$\langle a,b \mid [b,a,a,b]=[b,a,a,a,a]=0,\ [b,a,b,b]=[b,a,a,a],\ \text{class } 5\rangle, \tag{7.154}$$

$$\langle a,b \mid [b,a,a,b]=[b,a,a,a,a]=0,\ [b,a,b,b]=[b,a,a,a]+[b,a,a,a,b],\ \text{class } 5\rangle. \tag{7.155}$$

And if $\lambda=1$ then replacing b by $-b$ changes μ to $-\mu$ so we get $(p+1)/2$ diœerent algebras [DW.180]

$$\begin{aligned}\langle a,b \mid [b,a,a,a,a] &= 0,\ [b,a,a,b]=[b,a,a,a,b], \qquad (7.156)\\ [b,a,b,b] &= [b,a,a,a]+\mu[b,a,a,a,b],\ \text{class } 5\rangle\ (\mu=0,1,2,\ldots,(p-1)/2).\end{aligned}$$

We now consider the case when $p=1 \bmod 4$. In this case -1 is a square, say $-1=\eta^2$. Then replacing a by ηa we have

$$\begin{aligned}[b,a,a,b] &\in L^5,\\ [b,a,b,b] &= -[b,a,a,a] \bmod L^5.\end{aligned}$$

As above we consider all possible pairs of generators a',b' such that

$$\begin{aligned}[b',a',a',b'] &\in L^5,\\ [b',a',b',b'] &= -[b',a',a',a'] \bmod L^5.\end{aligned}$$

We see that

$$\begin{aligned}a' &= \alpha a+\beta b \bmod L^2,\\ b' &= \pm(\beta a+\alpha b) \bmod L^2,\end{aligned}$$

for some α,β with $\alpha^2-\beta^2\neq 0$. We then obtain

$$\begin{aligned}[b',a',b',b']+[b',a',a',a'] &= \pm\left((\alpha^4-\beta^4)([b,a,b,b]+[b,a,a,a])+4\alpha\beta(\alpha^2-\beta^2)[b,a,a,b]\right),\\ [b',a',a',b'] &= \alpha\beta(\alpha^2-\beta^2)([b,a,b,b]+[b,a,a,a])+(\alpha^4-\beta^4)[b,a,a,b],\\ [b',a',a',a',a'] &= \pm\left(\alpha(\alpha^2-\beta^2)^2[b,a,a,a,a]+\beta(\alpha^2-\beta^2)^2[b,a,a,a,b]\right),\\ [b',a',a',a',b'] &= \beta(\alpha^2-\beta^2)^2[b,a,a,a,a]+\alpha(\alpha^2-\beta^2)^2[b,a,a,a,b].\end{aligned}$$

As above, we have a non-trivial relation

$$\alpha[b,a,a,a,a]+\beta[b,a,a,a,b]=0,$$

and provided $\alpha \neq \pm\beta$ we can replace a, b by a', b', so that $[b', a', a', a', a'] = 0$ and we have one of the algebras 7.144 ˘ 7.156. So suppose that $\alpha = \pm\beta$. Replacing a by $-a$ if necessary we may suppose that $\alpha = -\beta$, so that we have the relation

$$[b, a, a, a, a] = [b, a, a, a, b].$$

We also have

$$\begin{aligned} [b, a, a, b] &= \lambda[b, a, a, a, a], \\ [b, a, b, b] + [b, a, a, a] &= \mu[b, a, a, a, a] \end{aligned}$$

for some λ, μ, where by scaling we may suppose that $\lambda = 0$ or 1. If $\lambda = 0$ then by scaling we may suppose that $\mu = 0$ or 1, and this gives (for $p = 1 \bmod 4$) [DW.183, and possibly DW.184 for some p]

$$\langle a, b \,|\, [b, a, a, b] = 0,\ [b, a, a, a, a] = [b, a, a, a, b], [b, a, b, b] = -[b, a, a, a],\ \text{class } 5\rangle, \tag{7.157}$$

$$\begin{aligned} \langle a, b \,|\, [b, a, a, b] &= 0,\ [b, a, a, a, a] = [b, a, a, a, b] \\ [b, a, b, b] &= -[b, a, a, a] + [b, a, a, a, a],\ \text{class } 5\rangle. \end{aligned} \tag{7.158}$$

If $\lambda = 1$, then once again we consider possible generating pairs a', b' such that

$$\begin{aligned} [b', a', a', a', a'] &= [b', a', a', a', b'], \\ [b', a', a', b'] &\in L^5, \\ [b', a', b', b'] + [b', a', a', a'] &\in L^5, \end{aligned}$$

and the calculation given above shows that we must have

$$\begin{aligned} a' &= \alpha a + \beta b \bmod L^2, \\ b' &= \beta a + \alpha b \bmod L^2. \end{aligned}$$

If $\mu \neq \pm 2$, and $\mu^2 - 4$ is a square, then we can ønd α, β with $\alpha \neq \pm\beta$ such that $[b', a', a', b'] = 0$, and we are back to 7.157 or 7.158. If $\mu = \pm 2$, or $\mu^2 - 4$ is not a square, then we need

$$\alpha\beta\mu + \alpha^2 + \beta^2 = (\alpha + \beta)(\alpha^2 - \beta^2)$$

to ensure that $[b', a', a', b'] = [b', a', a', a', a']$. But note that the left-hand side of this equation is homogeneous of degree 2 in α, β, whereas the right-hand side is homogeneous of degree 3, so that we can satisfy this equation for any given ratio α/β. Assuming that this equation holds, we have

$$[b', a', b', b'] + [b', a', a', a'] = \mu'[b', a', a', a', a']$$

where

$$\mu' = \frac{(\alpha^2+\beta^2)\mu + 4\alpha\beta}{\alpha\beta\mu + \alpha^2 + \beta^2}.$$

If $\mu = \pm 2$ then $\mu' = \mu$. So assume that $\mu^2 - 4$ is not a square. We show that as α/β takes all ratios other than ± 1, then μ' takes on all possible values such that $\mu'^2 - 4$ is not a square. Note that the value of μ' depends only on the ratio α/β (with the ratio 0 giving the same value as ∞). It is straightforward to show that

$$\frac{(x^2+1)\mu + 4x}{x\mu + x^2 + 1} = \frac{(y^2+1)\mu + 4y}{y\mu + y^2 + 1}$$

if and only if $\mu = \pm 2$ or $x = y$ or $xy = 1$. So if $\mu^2 - 4$ is not a square then μ' takes $(p+3)/2$ diœerent values as α/β takes all possible ratios. (Note that if $\mu^2 - 4$ is a square then one of these values is ∞.) However, if $\alpha/\beta = 1$ then $\mu' = 2$ and if $\alpha/\beta = -1$ then $\mu' = -2$. So as α/β takes all ratios other than ± 1, μ' takes on $(p-1)/2$ diœerent values. Finally note that

$$\left(\frac{(x^2+1)\mu + 4x}{x\mu + x^2 + 1}\right)^2 - 4 = \frac{(x^2-1)^2(\mu^2-4)}{(x\mu + x^2 + 1)^2}$$

so that $\mu'^2 - 4$ is a square if and only if $\mu^2 - 4$ is a square. This implies that $\mu \sim \mu'$ (provided $\mu' \neq \infty$) gives an equivalence relation with four equivalence classes: $\{2\}$, $\{-2\}$, $\{\mu \,|\, \mu^2 - 4 \text{ not a square}\}$, $\{\mu \,|\, \mu^2 - 4 \text{ a square}, \ \mu \neq \pm 2\}$ of sizes $1, 1, (p-1)/2, (p-3)/2$ respectively. So we obtain three further algebras (for $p = 1 \bmod 4$)

$$\begin{aligned} \langle a, b \,|\, [b,a,a,b] &= [b,a,a,a,a] = [b,a,a,a,b] \\ [b,a,b,b] &= -[b,a,a,a] + \mu[b,a,a,a,a], \text{ class } 5\rangle \ (\mu = \pm 2). \end{aligned} \qquad (7.159)$$

$$\begin{aligned} \langle a, b \,|\, [b,a,a,b] &= [b,a,a,a,a] = [b,a,a,a,b] \\ [b,a,b,b] &= -[b,a,a,a] + \mu[b,a,a,a,a], \text{ class } 5\rangle \ (\mu^2 - 4 \text{ not a square}). \end{aligned} \qquad (7.160)$$

The two algebras 7.159 appear to be missing from David Wilkinson's list, and 7.160 is probably DW.185, though I can't get it to match up exactly.

Our complete analysis of generating pairs a', b' such that $a' + L^5, b' + L^5$ satsify the same relations as $a + L^5, b + L^5$ show that these algebras are all distinct.

5.6.6 case 6

Let L be a descendant of 6.29. Then L is generated by a, b, L/L^4 is free of class 3, L^4 is spanned modulo L^5 by $[b,a,b,b]$ and L^5 is spanned by $[b,a,a,a,a]$ and $[b,a,a,a,b]$. The commutator $[b,a,a,b]$ lies in L^5, and $[b,a,b,b] = \omega[b,a,a,a]$ modulo L^5. Since L^5 has dimension 1, $[b,a,a,a,a]$ and $[b,a,a,a,b]$ must be linearly dependant.

In a similar way to the last section we consider all possible pairs of generators a', b' such that

$$\begin{aligned}[b',a',a',b'] &\in L^5,\\ [b',a',b',b'] &= \omega[b',a',a',a'] \bmod L^5.\end{aligned}$$

It is straightforward to show that

$$\begin{aligned}a' &= \alpha a + \beta b \bmod L^2,\\ b' &= \pm(-\omega\beta a + \alpha b) \bmod L^2,\end{aligned}$$

for some α, β with $\alpha^2 + \omega\beta^2 \neq 0$. Then

$$\begin{aligned}[b',a',b',b'] - \omega[b',a',a',a'] &= \pm\left((\alpha^4 - \omega^2\beta^4)([b,a,b,b] - \omega[b,a,a,a]) - 4\alpha\beta\omega(\alpha^2+\omega\beta^2)[b,a,a,b]\right),\\ [b',a',a',b'] &= \alpha\beta(\alpha^2+\omega\beta^2)([b,a,b,b] - \omega[b,a,a,a]) + (\alpha^4 - \omega^2\beta^4)[b,a,a,b],\\ [b',a',a',a',a'] &= \pm\left(\alpha(\alpha^2+\omega\beta^2)^2[b,a,a,a,a] + \beta(\alpha^2+\omega\beta^2)^2[b,a,a,a,b]\right),\\ [b',a',a',a',b'] &= -\beta\omega(\alpha^2+\omega\beta^2)^2[b,a,a,a,a] + \alpha(\alpha^2+\omega\beta^2)^2[b,a,a,a,b].\end{aligned}$$

So if the dependance relation between $[b,a,a,a,a]$ and $[b,a,a,a,b]$ is

$$\alpha[b,a,a,a,a] + \beta[b,a,a,a,b] = 0,$$

then provided $\alpha^2 + \omega\beta^2 \neq 0$ we can replace a, b by a', b', and then $[b,a,a,a,a] = 0$, and L^5 is spanned by $[b,a,a,a,b]$. (Note that $\alpha^2 + \omega\beta^2 = 0$ can only occur when $p = 3 \bmod 4$. We will deal with this situation later.) We have $[b,a,a,b] = \lambda[b,a,a,a,b]$ (where by scaling we can take $\lambda = 0$ or 1), and we have

$$[b,a,b,b] = \omega[b,a,a,a] + \mu[b,a,a,a,b]$$

for some μ.

Suppose for the moment that $[b,a,a,b] = [b,a,a,a,a] = 0$. If let $a' = \nu a$, $b' = \nu b$, then

$$[b',a',b',b'] = \nu^4[b,a,b,b] = \omega[b',a',a',a'] + \mu\nu^{-1}[b',a',a',a',b'],$$

so by scaling we may suppose that $\mu = 0$ or 1. So we have [DW.176, DW.175]

$$\langle a,b \mid [b,a,a,b] = [b,a,a,a,a] = 0,\ [b,a,b,b] = \omega[b,a,a,a],\ \text{class } 5\rangle, \tag{7.161}$$

$$\langle a,b \mid [b,a,a,b] = [b,a,a,a,a] = 0,\ [b,a,b,b] = \omega[b,a,a,a] + [b,a,a,a,b],\ \text{class } 5\rangle. \tag{7.162}$$

Next suppose that $[b,a,a,a,a] = 0$, $[b,a,a,b] = [b,a,a,a,b]$. As above we have

$$[b,a,b,b] = \omega[b,a,a,a] + \mu[b,a,a,a,b],$$

and replacing b by $-b$ we can alter the parameter μ to $-\mu$. So we get $(p+1)/2$ diœerent algebras [DW.174]

$$\begin{aligned} \langle a,b \mid [b,a,a,a,a] &= 0,\ [b,a,a,b]=[b,a,a,a,b], \\ [b,a,b,b] &= \omega[b,a,b,b]+\mu[b,a,a,a,b],\ \text{class } 5\rangle\ (\mu=0,1,2,\dots,(p-1)/2). \end{aligned} \tag{7.163}$$

Now consider the case when $p=3 \bmod 4$. In this case -1 is not a square, and we can replace ω by -1. The relation

$$\alpha[b,a,a,a,a]+\beta[b,a,a,a,b]=0$$

can be dealt with just as above, except in the case $\alpha=\pm\beta$. So (replacing a by $-a$ if necessary) we suppose that

$$[b,a,a,a,a]=[b,a,a,a,b].$$

We then have$[b,a,a,b]=\lambda[b,a,a,a,a]$ (where by scaling we can take $\lambda=0$ or 1), and

$$[b,a,b,b]=-[b,a,a,a]+\mu[b,a,a,a,a]$$

for some μ.

If $[b,a,a,b]=0$ then by scaling we may suppose that $\mu=0$ or 1. This gives (for $p=3 \bmod 4$) [DW.177, DW.179]

$$\langle a,b \mid [b,a,a,b]=0,\ [b,a,a,a,a]=[b,a,a,a,b],[b,a,b,b]=-[b,a,a,a],\ \text{class } 5\rangle, \tag{7.157A}$$

$$\begin{aligned} \langle a,b \mid [b,a,a,b] &= 0,\ [b,a,a,a,a]=[b,a,a,a,b] \\ [b,a,b,b] &= -[b,a,a,a]+[b,a,a,a,a],\ \text{class } 5\rangle. \end{aligned} \tag{7.158A}$$

Note that these algebras have the same presentations as 7.157 and 7.158, though these are for $p=3 \bmod 4$ and are descendants of 6.29, whereas 7.157 and 7.158 are for $p=1 \bmod 4$ and are descendants of 6.28.

Now suppose that $[b,a,a,b]=[b,a,a,a,a]$ and that

$$[b,a,b,b]+[b,a,a,a]=\mu[b,a,a,a,a].$$

As above we consider generating pairs a',b' satisfying

$$\begin{aligned} [b',a',a',b'] &\in L^5, \\ [b',a',b',b'] &= \omega[b',a',a',a'] \bmod L^5. \end{aligned}$$

and also satisfying

$$[b',a',a',a',a']=[b',a',a',a',b']$$

The calculations above show that

$$\begin{aligned} a' &= \alpha a + \beta b \bmod L^2, \\ b' &= \beta a + \alpha b \bmod L^2, \end{aligned}$$

for some α, β with $\alpha^2 \neq \beta^2$. Then

$$\begin{aligned} [b',a',b',b'] + [b',a',a',a'] &= (\alpha^4-\beta^4)([b,a,b,b]+[b,a,a,a]) + 4\alpha\beta(\alpha^2-\beta^2)[b,a,a,b], \\ [b',a',a',b'] &= \alpha\beta(\alpha^2-\beta^2)([b,a,b,b]+[b,a,a,a]) + (\alpha^4-\beta^4)[b,a,a,b], \\ [b',a',a',a',a'] &= \alpha(\alpha^2-\beta^2)^2[b,a,a,a,a] + \beta(\alpha^2-\beta^2)^2[b,a,a,a,b], \\ [b',a',a',a',b'] &= \beta(\alpha^2-\beta^2)^2[b,a,a,a,a] + \alpha(\alpha^2-\beta^2)^2[b,a,a,a,b]. \end{aligned}$$

If $\mu \neq \pm 2$, and $\mu^2 - 4$ is a square, then just as in the case when $p = 1 \bmod 4$ in descendants of 6.28, we can ønd α, β with $\alpha \neq \pm\beta$ such that $[b',a',a',b'] = 0$, and we are back to 7.157A or 7.158A. If $\mu = \pm 2$, or $\mu^2 - 4$ is not a square, then we need

$$\alpha\beta\mu + \alpha^2 + \beta^2 = (\alpha+\beta)(\alpha^2-\beta^2)$$

to ensure that $[b',a',a',b'] = [b',a',a',a',a']$. Assuming that this equation holds, we have

$$[b',a',b',b'] + [b',a',a',a'] = \mu'[b',a',a',a',a']$$

where

$$\mu' = \frac{(\alpha^2+\beta^2)\mu + 4\alpha\beta}{\alpha\beta\mu + \alpha^2 + \beta^2}.$$

If $\mu = \pm 2$ then $\mu' = \mu$ for all α, β. But if $\mu \neq \pm 2$ and $\mu^2 - 4$ is not a square, then since $p = 3 \bmod 4$, $4 - \mu^2$ is a square, and we can ønd α, β, with $\alpha \neq \pm\beta$, so that $\mu' = 0$. So we have three algebras (for $p = 3 \bmod 4$) [DW.178, and two algebras missing from David Wilkinson's list]

$$\begin{aligned} \langle a, b \mid [b,a,a,b] &= [b,a,a,a,a] = [b,a,a,a,b] \\ [b,a,b,b] &= -[b,a,a,a], \text{ class } 5\rangle, \end{aligned} \tag{7.164}$$

$$\begin{aligned} \langle a, b \mid [b,a,a,b] &= [b,a,a,a,a] = [b,a,a,a,b] \\ [b,a,b,b] &= -[b,a,a,a] + \lambda[b,a,a,a,a], \text{ class } 5\rangle\ (\lambda = \pm 2). \end{aligned} \tag{7.159A}$$

Note that these last two algebras have the same presentations as those given by 7.159, though these are for $p = 3 \bmod 4$ and are descendants of 6.29, whereas the algebras given by 7.159 are for $p = 1 \bmod 4$ and are descendants of 6.28.

Once again, our complete analysis of generating pairs a', b' such that $a' + L^5, b' + L^5$ satsify the same relations as $a + L^5, b + L^5$ show that these algebras are all distinct.

5.6.7 case 7

Let L be a descendant of 6.30. Then L is generated by a, b, and has class 6. For $i = 2, 3, 4, 5, 6$, L^i is spanned by $[b, a,_{i-1} b]$ modulo L^{i+1}. We have $[b, a, b, b, a] = \lambda[b, a, b, b, b, b]$ and $[b, a, a] = \mu[b, a, b, b, b, b]$ for some λ, μ. By scaling we can take $\lambda = 0$ or 1. If $\lambda = 0$ then we can take $\mu = 0$ or 1 by scaling. This gives [DW.190, DW.191]

$$\langle a, b \mid [b, a, a] = [b, a, b, b, a] = 0, \text{ class } 6\rangle, \tag{7.165}$$

$$\langle a, b \mid [b, a, a] = [b, a, b, b, b, b], [b, a, b, b, a] = 0, \text{ class } 6\rangle. \tag{7.166}$$

If $\lambda = 1$ we let $a' = a + (\mu/2)[b, a, b]$. Then

$$\begin{aligned}
[b, a'] &= [b, a] - (\mu/2)[b, a, b, b], \\
[b, a', a'] &= [b, a, a] - \mu[b, a, b, b, b, b] = 0, \\
[b, a', b] &= [b, a, b] - (\mu/2)[b, a, b, b, b], \\
[b, a', b, b] &= [b, a, b, b] - (\mu/2)[b, a, b, b, b, b], \\
[b, a', b, b, b] &= [b, a, b, b, b], \\
[b, a', b, b, b, b] &= [b, a, b, b, b, b], \\
[b, a', b, b, a'] &= [b, a, b, b, a] = [b, a, b, b, b, b].
\end{aligned}$$

So replacing a by a' we have [DW.192]

$$\langle a, b \mid [b, a, a] = 0, [b, a, b, b, a] = [b, a, b, b, b, b], \text{ class } 6\rangle. \tag{7.167}$$

In 7.165 the centralizer of the derived algebra has dimension 6, in 7.166 it has dimension 5, and in 7.167 it has dimension 3. So these algebras are all diœerent.

5.6.8 case 8

Let L be a descendant of 6.31. Then L is generated by a, b, and has class 6. And, as in the case above, for $i = 2, 3, 4, 5, 6$, L^i is spanned by $[b, a,_{i-1} b]$ modulo L^{i+1}. We have $[b, a, b, b, a] = \lambda[b, a, b, b, b, b]$ and $[b, a, a] = [b, a, b, b, b] + \mu[b, a, b, b, b, b]$ for some λ, μ. By scaling we can take $\lambda = 0$ or 1. If $\lambda = 0$ then by scaling we can take $\mu = 0$ or 1. This gives [DW.194, DW.193]

$$\langle a, b \mid [b, a, a] = [b, a, b, b, b], [b, a, b, b, a] = 0, \text{ class } 6\rangle, \tag{7.168}$$

$$\langle a, b \mid [b, a, a] = [b, a, b, b, b] + [b, a, b, b, b, b], [b, a, b, b, a] = 0, \text{ class } 6\rangle. \tag{7.169}$$

If $\lambda = 1$ then, just as in the last case we let $a' = a + (\mu/2)[b, a, b]$. Then

$$\begin{aligned}
[b, a'] &= [b, a] - (\mu/2)[b, a, b, b], \\
[b, a', a'] &= [b, a, a] - \mu[b, a, b, b, b, b] = [b, a, b, b, b], \\
[b, a', b] &= [b, a, b] - (\mu/2)[b, a, b, b, b],
\end{aligned}$$

$$\begin{aligned}
[b,a',b,b] &= [b,a,b,b]-(\mu/2)[b,a,b,b,b,b],\\
[b,a',b,b,b] &= [b,a,b,b,b],\\
[b,a',b,b,b,b] &= [b,a,b,b,b,b],\\
[b,a',b,b,a'] &= [b,a,b,b,a]=[b,a,b,b,b,b].
\end{aligned}$$

So replacing a by a' we have [DW.195]

$$\langle a,b \mid [b,a,a]=[b,a,b,b,b],\ [b,a,b,b,a]=[b,a,b,b,b,b],\ \text{class } 6\rangle. \tag{7.170}$$

In 7.168 and 7.69 the centralizer of the derived algebra has dimension 5, but in 7.170 it has dimension 3. To show that 7.168 and 7.169 are distinct we consider all possible pairs a', b' of generators of 7.168 with the property that $a'+L^6, b'+L^6$ satisfy the same relations as $a+L^6, b+L^6$. If easy to see that

$$\begin{aligned}
a' &= \alpha^3 a+g,\\
b' &= \beta a+\alpha b+h
\end{aligned}$$

for some α,β with $\alpha\neq 0$ and for some $g,h\in L^2$. And then it is straightforward to show that

$$[b',a',a']=[b',a',b',b',b'],\ [b',a',b',b',a']=0,$$

so 7.168 is not isomorphic to 7.169.

5.6.9 case 9

Let L be a descendant of 6.33. Then L is generated by a,b, and has class 6. And, as in the case above, for $i=2,3,4,5,6$, L^i is spanned by $[b,a,_{i-1}b]$ modulo L^{i+1}. We have $[b,a,b,b,a]=\lambda[b,a,b,b,b,b]$ and $[b,a,a]=[b,a,b,b]+\mu[b,a,b,b,b,b]$ for some λ,μ. First consider the case when $\lambda\neq 1$. Let $a'=a+\alpha[b,a,b]$. Then

$$\begin{aligned}
[b,a'] &= [b,a]-\alpha[b,a,b,b],\\
[b,a',a'] &= [b,a,a]+2\alpha(1-\lambda)[b,a,b,b,b,b]\\
&= [b,a,b,b]+(2\alpha-2\alpha\lambda+\mu)[b,a,b,b,b,b],\\
[b,a',b] &= [b,a,b]-\alpha[b,a,b,b,b],\\
[b,a',b,b] &= [b,a,b,b]-\alpha[b,a,b,b,b,b],\\
[b,a',b,b,b] &= [b,a,b,b,b],\\
[b,a',b,b,b,b] &= [b,a,b,b,b,b],\\
[b,a',b,b,a'] &= [b,a,b,b,a]=\lambda[b,a,b,b,b,b].
\end{aligned}$$

So if we take $\alpha=\mu(2-2\lambda)^{-1}$ and replace a by a' we have $p-1$ algebras [DW.196]

$$\langle a,b \mid [b,a,a]=[b,a,b,b],\ [b,a,b,b,a]=\lambda[b,a,b,b,b,b],\ \text{class } 6\rangle\ (\lambda\neq 1). \tag{7.171}$$

Next consider the case when $\lambda = 1$. If we let $a' = \alpha^2 a$, $b' = \alpha b$ for some $\alpha \neq 0$, then

$$\begin{aligned} [b', a', b', b', a'] &= \alpha^7[b, a, b, b, a] = \alpha^7[b, a, b, b, b, b] = [b', a', b', b', b', b'], \\ [b', a', a'] &= \alpha^5[b, a, a] = \alpha^5[b, a, b, b] + \alpha^5\mu[b, a, b, b, b, b] \\ &= [b', a', b', b'] + \alpha^{-2}\mu[b', a', b', b', b', b']. \end{aligned}$$

So we may assume that $\mu = 0, 1$, or ω, which gives [DW.197, DW.198, DW.199]

$$\begin{aligned} \langle a, b \mid [b, a, a] &= [b, a, b, b] + \mu[b, a, b, b, b, b], \\ [b, a, b, b, a] &= [b, a, b, b, b, b], \text{ class } 6\rangle \ (\mu = 0, 1, \omega). \end{aligned} \quad (7.172)$$

As in the previous case we show that these algebras are distinct by considering all possible generating pairs a', b' such that $a' + L^6, b' + L^6$ satisfy the same relations as $a + L^6, b + L^6$. It is straightforward to see that

$$\begin{aligned} a' &= \alpha^2 a + g, \\ b' &= \alpha b + h \end{aligned}$$

for some α with $\alpha \neq 0$ and for some $g, h \in L^2$. And then it is straightforward to show that

$$\begin{aligned} [b', a', b', b', b', b'] &= \alpha^7[b, a, b, b, b, b], \\ [b', a', b', b', a'] &= \alpha^7[b, a, b, b, a], \end{aligned}$$

so dißerent values of λ give dißerent algebras. When $\lambda = 1$ then we can show that

$$[b', a', a'] - [b', a', b', b'] = \alpha^5([b, a, a] - [b, a, b, b]),$$

so $\mu = 0, 1$, or ω give dißerent algebras.